UNIVERSITY OF TECHNOLOGY AND ARTS OF BYUMBA (UTAB)

EDUCATION ; SS MDS & AEMRE FACULTIES

FUNDAMENTALS OF MATHEMATICS MODULE
(LINEAR ALGEBRA & CALCULUS UNITS)

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The module comprises in chapter **I** the basic *algebra* review aiming to assess students skills from their prior knowledge and at the same time providing them with the content they need to remediate those skills.
Thus we will be able to tackle smoothly the chapters of **Part I:**
Linear Equations and Graphs; **Functions-Graphs**; **Systems of Linear Equations**-**Matrices**. **Determinants.**

The Part II is allocated to the **Calculus** and **includes** the following chapters:
Limits and the Derivative; Additional Derivative Topics; Graphing and Optimization and Integration

The **discussion** and **exploration** of problems lead into new concepts and help students gain better insight into the mathematical concepts through thought-provoking questions that are effective in classroom. Thus Teacher will be able to **easily craft homework assignments** that best meet the needs of students by taking advantage of the variety of types and difficulty levels of the exercises set at the end of **each Chapter**.

PART I: LINEAR ALGEBRA.

The rules for manipulating and reasoning with symbols in algebra depend, in large measure, **on properties of the real numbers**. In this section we look at some of the important properties of this number system. To make our discussions here and elsewhere in the module clearer and more precise, we occasionally make use of simple *set* concepts and notation.

CHAPTER I: SOME PREREQUISITE TOPICS

I.1: SET OF REAL NUMBERS.

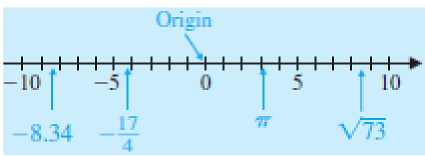
Informally, **a real number is any number that has a decimal representation**. **Table 1** describes the set of real numbers and some of its important subsets. **Figure 1** illustrates how these sets of numbers are related. **The set of integers** contains all the natural numbers and something else—their negatives and 0. The set of rational numbers contains all the integers and something else—non integer ratios of integers. And the set of real numbers contains all the rational numbers and something else—irrational numbers.

TABLE 1: REAL NUMBERS

SYMBOL	NAME	DESCRIPTION	EXAMPLES
N	**NATURAL NUMBERS.**	**COUNTING NUMBERS** (ALSO CALLED POSITIVE INTEGERS)	**1, 2, 3, ……**
Z	**INTEGERS NATURAL NUMBERS**	**THEIR NEGATIVES, AND 0**	**…, -2, -1, 0, 1, 2, …..**
Q	**RATIONAL NUMBERS**	**RATIONAL NUMBERS NUMBERS THAT CAN BE REPRESENTED AS *A/B*, WHERE *A* AND *B* ARE INTEGERS AND *B*$\neq$ 0 DECIMAL REPRESENTATIONS ARE REPEATING OR TERMINATING**	**-4,0,1,25,$\frac{-3}{5}$,3.67, - 0.333,5.272727**
I	**IRRATIONAL NUMBERS**	**NUMBERS THAT CAN BE REPRESENTED AS NONREPEATING AND NONTERMINATING DECIMAL NUMBERS**	**$\sqrt{2}$, π, $\sqrt[3]{7}$,1.414213,2.71828182**
R	**REAL NUMBERS**	**RATIONAL AND IRRATIONAL NUMBERS**	

I.2: REAL NUMBER LINE

A one-to-one correspondence exists between **the set of real numbers** and **the set of points on a line**. That is, each real number corresponds to exactly one point, and each point corresponds to exactly one real number. A line with a real number associated with each point, and vice versa, as shown in **figure 1**, is called a **real number line**, or simply a **real line**. **Each number** associated with a point is called **the coordinate of the point**:



I.3.BASIC REAL NUMBER PROPERTIES

In mathematics, we often have to perform some or all of the four major operations of arithmetic on real numbers. These are addition (**+**), subtraction (**−**), multiplication (**×**) and division (**÷**). There are simple rules and conventions which we need to observe:

(a) Operations within brackets are performed first.

(b) If there are no brackets to indicate priority, then multiplication and division take precedence over addition and subtraction.

(c) Addition and subtraction are performed in their order of appearance.

(d) Multiplication and division are performed in their order of appearance.

(e) A number of additions can be performed in any order. For any real numbers a, b, c $\in$ **R*, we have*** **:**

$$a + (b + c) = (a + b) + c \text{ and } a + b = b + a.$$

Example 1.3.1. We have $-3 \times 4 - 5 + (-3) = -(3 \times 4) - 5 + (-3) = -12 - 5 - 3 = -20$. Note that we have recognized that 3×4 takes precedence over the − signs.

Example 1.3.2. We have

$21 + 32 \div (-4) + (-6) = 21 + (32 \div (-4)) + (-6) = 21 + (-8) + (-6)$

$= 21 - 8 - 6 = (21 - 8) - 6 = 13 - 6 = 7.$

Note that we have recognized that $32 \div (-4)$ takes precedence over the + signs, and that $21 - 8$ takes precedence over the following − sign. Note that we have recognized that $32 \div (-4)$ takes precedence over the + signs, and that $21 - 8$ takes precedence over the following − sign.

Example 1.3.3. We have $(366\div(-6)-(-6))\div(-11) = ((-61)-(-6))\div(-11) = (-55)\div(-11) = 5$.Note that the division by -11 is performed last because of brackets.

Example 1.3.4. We have $720 \div (-9) \div 4 \times (-2) = (-80) \div 4 \times (-2) = (-20) \times (-2) = 40$.

Example 1.3.5. Convince yourself that $(76 \div 2 - (-2) \times 9 + 4 \times 8) \div 4 \div 2 - (10 - 3 \times 3) - 6 = 4$. Another operation on real numbers that we perform frequently is taking square roots:

Suppose that $a \geq 0$. We say that x is a square root of a if $x^2 = a$.

Remarks. (1) If $a > 0$, then there are two square roots of a. We denote by

$\sqrt{a}$ *the* positive square root of a, and by $-\sqrt{a}$ *the* negative square root of a.

(2) If $a = 0$, then there is only one square root of a. We have

$\sqrt{0} = 0$.

(3) Note that square root of a is not defined when $a < 0$. If x is a real number, then $x^2 \geq 0$ and

so cannot be equal to any real negative number a.

Example 1.3.6. We have $\sqrt{(76 \div 2 - (-2) \times 9 + 4 \times 8) \div 4 \div 2 - (10 - 3 \times 3) - 6} = 2.$

Example 1.2.7. We have $\sqrt{27} = 3 \times \sqrt{3}$. To see this, note that

$$(3 \times \sqrt{3})^2 = 3 \times \sqrt{3} \times 3 \times \sqrt{3} = 3 \times 3 \times \sqrt{3} \times \sqrt{3} = 3 \times 3 \times 3 = 27.$$

Example 1.3.7. We have $\sqrt{72} = \sqrt{2 \times 2 \times 2 \times 3 \times 3} = 2 \times 3 \times \sqrt{2} = 6 \times \sqrt{2}.$

1.4. DISTRIBUTIVE LAWS.

We now consider the distribution of multiplication inside brackets. For convenience, we usually suppress the multiplication sign ×, and write ab to denote the product $a \times b$. For every $a,b,c,d \in R$, we have

(a) $a(b + c) = ab + ac$;
(b) $(a + b)c = ac + bc$; and
(c) $(a + b)(c + d) = ac + ad + bc + bd$.

Special cases of part (c) above include the following two laws.

***LAWS ON SQUARES:**

For every $a,b \in R$, we have

(a) $(a + b)^2 = a^2 + 2ab+b^2$;
(b) $(a - b)^2 = a^2 - 2ab + b^2$; and
(c) $a^2 - b^2 = (a - b)(a + b)$

***LAWS ON CUBES**
(a) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$***; and***
(b) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Example **1.4.1.** Consider the expression $(2x+5)^2-(x+5)^2$. Using part (a) on the Laws on squares, we have

$(2x + 5)^2 = 4x^2 + 20x + 25$ and $(x + 5)^2 = x^2 + 10x + 25$. It follows that

$$(2x + 5)^2- (x + 5)^2 = (4x^2 + 20x + 25) - (x^2 + 10x + 25) = 4x^2 + 20x + 25$$
$$- x^2- 10x - 25 = 3x^2 + 10x.$$

Example **1.4.2**. Consider the expression $(x - y)(x + y - 2) + 2x$. Using an extended version of part (c) of the Distributive laws, we have

$(x - y)(x + y - 2) = x^2 + xy - 2x - xy - y^2 + 2y = x^2- 2x - y^2 + 2y.$

It follows that

$$(x - y)(x + y - 2) + 2x = (x^2- 2x - y^2 + 2y) + 2x = x^2- 2x - y^2 + 2y + 2x = x^2- y^2 + 2y.$$

Alternatively, we have

$$(x-y)(x+y-2)+2x=(x-y)((x+y)-2)+2x=(x-y)(x+y)-2(x-y)+2x$$
$$=x^2-y^2-(2x-2y)+2x=x^2-y^2-2x+2y+2x=x^2-y^2+2y.$$

Example 1.4.3. We have

$$(x+1)(x-2)(x+3)=(x^2-2x+x-2)(x+3)=(x^2-x-2)(x+3)$$
$$=x^3+3x^2-x^2-3x-2x-6=x^3+2x^2-5x-6.$$

Example 1.4.4 :

We have $(5x + 3)^2 - (2x - 3)^2 + (3x - 2)(3x + 2) = (25x^2 + 30x + 9) - (4x^2 - 12x + 9) + (9x^2 - 4)$

$= 25x^2 + 30x + 9 - 4x^2 + 12x - 9 + 9x^2 - 4 = 30x^2 + 42x - 4.$

1.5. ARITHMETIC OF FRACTIONS

Suppose that we wish to **add** or **substract** two fractions:
$\frac{a}{b}+\frac{c}{d}$ or $\frac{a}{b}-\frac{c}{d}$ where $a,b,c,d \in Z$ with $b \neq 0$ and $d \neq 0$. For convenience, we have relaxed the requirement that b and d are positive integers.

We have:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

In both cases, we first rewrite the fractions with a common denominator, and then perform addition or subtraction on the numerators. Where possible, we may also perform some cancellation to the answer.

Example 1.5.1. Following the rules precisely, we have

$$\frac{1}{3}+\frac{1}{6}=\frac{6}{18}+\frac{3}{18}=\frac{6+3}{18}=\frac{9}{18}=\frac{1}{2}.$$

However, we can somewhat simplify the argument by using the lowest common denominator instead of the product of the denominators, and obtain

$$\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{2+1}{6}=\frac{3}{6}=\frac{1}{2}.$$

The next few examples may involve ideas discussed in the previous sections. The reader is advised to try to identify the use of the various laws discussed earlier.

Example 1.5.2 Consider the expression

$$\frac{(x-4)^2}{(x+4)^2}-\frac{(x+2)^2}{(x+4)^2}.$$

Here the denominators are the same, so we need only perform subtraction on the numerators. We have

$$\frac{(x-4)^2}{(x+4)^2} - \frac{(x+2)^2}{(x+4)^2} = \frac{(x-4)^2-(x+2)^2}{(x+4)^2} = \frac{(x^2-8x+16)-(x^2+4x+4)}{(x+4)^2}$$
$$= \frac{x^2-8x+16-x^2-4x-4}{(x+4)^2} = \frac{12-12x}{(x+4)^2}.$$

Example 1.5.3. We have

$$\frac{3(x-1)}{x+1} + \frac{2(x+1)}{x-1} = \frac{3(x-1)^2}{(x+1)(x-1)} + \frac{2(x+1)^2}{(x+1)(x-1)} = \frac{3(x-1)^2+2(x+1)^2}{(x+1)(x-1)}$$
$$= \frac{3(x^2-2x+1)+2(x^2+2x+1)}{x^2-1} = \frac{(3x^2-6x+3)+(2x^2+4x+2)}{x^2-1}$$
$$= \frac{3x^2-6x+3+2x^2+4x+2}{x^2-1} = \frac{5x^2-2x+5}{x^2-1}.$$

Example 1.5.4. We have

$$\frac{x}{y} - \frac{x}{x+y} = \frac{x(x+y)}{y(x+y)} - \frac{yx}{y(x+y)} = \frac{x(x+y)-yx}{y(x+y)} = \frac{x^2+xy-yx}{y(x+y)} = \frac{x^2+xy-xy}{y(x+y)} = \frac{x^2}{y(x+y)}.$$

Example 1.5.5. We have

$$\frac{p}{p-q} + \frac{q}{q-p} = \frac{p}{p-q} + \frac{-q}{p-q} = \frac{p+(-q)}{p-q} = \frac{p-q}{p-q} = 1.$$

Note here that the two denominators are essentially the same, apart from a sign change. Changing the sign of both the numerator and denominator of one of the fractions has the effect of giving two fractions with the same denominator.

Example 1.5.6. We have

$$\frac{4}{a} - \frac{2}{a(a+2)} = \frac{4(a+2)}{a(a+2)} - \frac{2}{a(a+2)} = \frac{4(a+2)-2}{a(a+2)} = \frac{(4a+8)-2}{a(a+2)} = \frac{4a+8-2}{a(a+2)} = \frac{4a+6}{a(a+2)}.$$

Note here that the common denominator is not the product of the two denominators, since we have observed the common factor *a* in the two denominators. If we do not make this observation, then we have

$$\frac{4}{a} - \frac{2}{a(a+2)} = \frac{4a(a+2)}{a^2(a+2)} - \frac{2a}{a^2(a+2)} = \frac{4a(a+2)-2a}{a^2(a+2)} = \frac{(4a^2+8a)-2a}{a^2(a+2)}$$
$$= \frac{4a^2+8a-2a}{a^2(a+2)} = \frac{4a^2+6a}{a^2(a+2)} = \frac{a(4a+6)}{a^2(a+2)} = \frac{4a+6}{a(a+2)}.$$

Note that the common factor *a* is cancelled from the numerator and denominator in the last step. We still have the same answer, but a little extra work is required.

Suppose next that we wish **to multiply two fractions** and consider:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

where *a,b,c,d* ∈Z with b$\neq 0$ and $d \neq 0$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

We simply multiply the numerators and denominators separately. Where possible, we may alsoperform some cancellation to the answer.

Example 1.5.7. We have

$$\frac{h}{x^2}\left(1-\frac{h}{x+h}\right)=\frac{h}{x^2}\left(\frac{x+h}{x+h}-\frac{h}{x+h}\right)=\frac{h}{x^2}\times\frac{x+h-h}{x+h}=\frac{h}{x^2}\times\frac{x}{x+h}=\frac{hx}{x^2(x+h)}=\frac{h}{x(x+h)}.$$

Example 1.5.8. We have

$$\left(\frac{1}{x}+\frac{1}{y}\right)(x+y)=\left(\frac{y}{xy}+\frac{x}{xy}\right)(x+y)=\frac{y+x}{xy}\times(x+y)=\frac{x+y}{xy}\times\frac{x+y}{1}=\frac{(x+y)^2}{xy}.$$

Example 1.5.9. We have

$$\left(\frac{1}{x}-\frac{1}{y}\right)\frac{1}{x-y}=\left(\frac{y}{xy}-\frac{x}{xy}\right)\frac{1}{x-y}=\frac{y-x}{xy}\times\frac{1}{x-y}=\frac{y-x}{xy(x-y)}=\frac{-(x-y)}{xy(x-y)}=-\frac{1}{xy}.$$

Example 1.5.10. We have

$$\frac{b-c}{bc}\times\frac{b^2}{b^2-bc}=\frac{b^2(b-c)}{bc(b^2-bc)}=\frac{b^2(b-c)}{b^2c(b-c)}=\frac{1}{c}.$$

Example 1.5.11. We have

$$\frac{a^2}{a^2-1}\times\frac{a+1}{a}=\frac{a^2(a+1)}{a(a^2-1)}=\frac{a^2(a+1)}{a(a-1)(a+1)}=\frac{a}{a-1}.$$

Example 1.5.12. We have

$$\frac{x+y}{x^2-4y^2}\times\frac{6y-3x}{2x+2y}=\frac{(x+y)(6y-3x)}{(x^2-4y^2)(2x+2y)}=\frac{3(x+y)(2y-x)}{2(x-2y)(x+2y)(x+y)}=-\frac{3}{2(x+2y)}.$$

Suppose finally that we wish **to divide one fraction by another** and consider:

$$\frac{(a/b)}{(c/d)}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

where $a,b,c,d \in \mathbb{Z}$ with $b\neq 0$ and $d\neq 0$. In other words, we invert the divisor and then perform multiplication instead. Where possible, we may also perform some cancellation to the answer. **Note the special cases that**

$$\frac{(a/b)}{c}=\frac{a}{bc} \quad \text{and} \quad \frac{a}{(c/d)}=\frac{ad}{c}.$$

Example 1.5.13. We have

$$\begin{aligned}\left(1+\frac{1}{1+x}\right)\div\frac{4}{5(1+x)}&=\left(\frac{1+x}{1+x}+\frac{1}{1+x}\right)\div\frac{4}{5(1+x)}=\frac{1+x+1}{1+x}\div\frac{4}{5(1+x)}\\&=\frac{2+x}{1+x}\div\frac{4}{5(1+x)}=\frac{2+x}{1+x}\times\frac{5(1+x)}{4}\\&=\frac{5(2+x)(1+x)}{4(1+x)}=\frac{5(2+x)}{4}.\end{aligned}$$

Example 1.5.14. We have

$$\frac{6a}{a-5}\div\frac{a+5}{a^2-25}=\frac{6a}{a-5}\times\frac{a^2-25}{a+5}=\frac{6a(a^2-25)}{(a-5)(a+5)}=\frac{6a(a^2-25)}{a^2-25}=6a.$$

Example 1.5.15. We have

$$\frac{\frac{1}{x}-\frac{1}{y}}{x-y}=\left(\frac{1}{x}-\frac{1}{y}\right)\div(x-y)=\left(\frac{y}{xy}-\frac{x}{xy}\right)\times\frac{1}{x-y}=\frac{y-x}{xy}\times\frac{1}{x-y}=\frac{y-x}{xy(x-y)}=-\frac{1}{xy}.$$

Example 1.5.16. We have

$$\frac{x^2-y^2}{\frac{1}{x}+\frac{1}{y}}=(x^2-y^2)\div\left(\frac{1}{x}+\frac{1}{y}\right)=(x^2-y^2)\div\left(\frac{y}{xy}+\frac{x}{xy}\right)=(x^2-y^2)\div\frac{y+x}{xy}$$
$$=(x^2-y^2)\times\frac{xy}{x+y}=\frac{xy(x^2-y^2)}{x+y}=\frac{xy(x-y)(x+y)}{x+y}=xy(x-y).$$

1.1.5: FACTORIZATION

Very often, we have to handle mathematical expressions that can be simplified. We have seen a few instances of cancellation of common terms in the numerator and denominator of fractions. We now consider the question of factorization. This can be thought of as the reverse process of expansion. It is difficult, if not impossible, to write down rules for factorization. Instead, we shall look at a few examples, and illustrate some of the ideas.

Example 1.1.5.1. Consider the expression $x^3 - x$. First of all, we recognize that both terms x^3 and x have a factor x. Hence we can write $x^3 - x = x(x^2 - 1)$, using one of the Distributive laws. Next, we realize that we can apply one of the Laws on squares on the factor $x^2 - 1$. Hence

$$x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1).$$

Example 1.1.5.2. Consider the expression $a^4 - b^4$. Note that we can apply one of the Laws on squares to obtain $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$. We can again apply one of the Laws on squares on the factor $a^2 - b^2$. Hence $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2)$.

Example 1.1.5.3. Consider the expression $x^3 - 64$. Note that $64 = 4^3$. Applying one of the Laws on cubes, we obtain $x^3 - 64 = (x - 4)(x^2 + 4x + 16)$.

Example 1.1.5.4. Consider the expression $m^2 - n^2 + 4m + 4n$. We may write

$$m^2 - n^2 + 4m + 4n = m^2 + 4m + 4n - n^2 = m(m + 4) + n(4 - n),$$

and this does not lead anywhere. However, we may recognize that

$$m^2 - n^2 + 4m + 4n = (m - n)(m + n) + 4(m + n) = (m - n + 4)(m + n),$$

and this gives a good factorization.

Example 1.1.5.5. We have

$$\begin{aligned} x^3-2x^2-4x+8 &= (x^3-2x^2)-(4x-8) = x^2(x-2)-4(x-2) \\ &= (x^2-4)(x-2) = (x-2)(x+2)(x-2) = (x-2)^2(x+2). \end{aligned}$$

Example 1.1.5.6. We have

$$\begin{aligned} \frac{a+1}{a^2-a}-\frac{a-1}{a^2+a} &= \frac{a+1}{a(a-1)}-\frac{a-1}{a(a+1)} = \frac{(a+1)^2}{a(a-1)(a+1)}-\frac{(a-1)^2}{a(a-1)(a+1)} = \frac{(a+1)^2-(a-1)^2}{a(a-1)(a+1)} \\ &= \frac{((a+1)-(a-1))((a+1)+(a-1))}{a(a-1)(a+1)} = \frac{(a+1-a+1)(a+1+a-1)}{a(a-1)(a+1)} \\ &= \frac{4a}{a(a-1)(a+1)} = \frac{4}{(a-1)(a+1)}. \end{aligned}$$

Example 1.5.7. We have

$$\begin{aligned} \frac{2}{x^2-1}-\frac{1}{x^2-x}+\frac{x-1}{x^2+x} &= \frac{2}{(x-1)(x+1)}-\frac{1}{x(x-1)}+\frac{x-1}{x(x+1)} \\ &= \frac{2x}{x(x-1)(x+1)}-\frac{x+1}{x(x-1)(x+1)}+\frac{(x-1)^2}{x(x-1)(x+1)} \\ &= \frac{2x-(x+1)+(x-1)^2}{x(x-1)(x+1)} = \frac{2x-x-1+x^2-2x+1}{x(x-1)(x+1)} \\ &= \frac{x^2-x}{x(x-1)(x+1)} = \frac{x(x-1)}{x(x-1)(x+1)} = \frac{1}{x+1}. \end{aligned}$$

ACTIVITIES OF CHAPTER I

ACTVITIES I.1-Find the precise value of each of the following expressions:

a) $5+4\times 3\div 2-1$

b) $(1+2)\times 3-4\div 5$

c) $(54\div 3+18\times 2)\div 9$

d) $4+2-4+5\times(-2)\times(1+3)$

e) $2+5\times(-1)-(2+3)\times 4\div 10+4-(3-5)$

f) $((4+2)\times 3+1)\times 5+10\div 2$

g) $\sqrt{(-4)\times(2-11)}$

h) $-\sqrt{5^2+12^2}$

i) $\sqrt{5\times 5-4\times 4}-\sqrt{3\times 3-2\times 2-(-1)\times(-1)}$

ACTVITIES I.2-Expand each of the following expressions:

a) $(4x+3)(5x-2)$

b) $(4x+3)^2+(5x-2)^2$

c) $(4x+3)^2-(5x-2)^2$

d) $(7x-2)^2+(4x+5)^2$

e) $(x+y)(x-2y)$

f) $(x+2y+3)(2x-y-1)$

g) $(x+2y)(x-2y)^2$

h) $(x+2y)^2(x-2y)^2$

ACTVITIES I.3: Rewrite each of the following expressions, showing all the steps of your argument carefully:

a) $\frac{3}{4}+\frac{2}{3}$

b) $\frac{5}{6}-\frac{1}{12}$

c) $\frac{5x}{x+2}+\frac{3x}{x+4}$

d) $\frac{3}{x-1}-\frac{3}{x+1}$

e) $\frac{5}{x}+\frac{3}{x(x+1)}$

f) $\frac{(x+y)^2}{x^2}-\frac{(x-y)^2}{y^2}$

ACTVITIES I.4: Rewrite each of the following expressions, showing all the steps of your argument carefully:

a) $\frac{2+3}{4+5}\times\frac{6+7}{8+9}$

b) $\frac{2+3}{4+5}\div\frac{6+7}{8+9}$

c) $\left(\frac{1}{2}+\frac{1}{3}\right)\div\left(\frac{3}{4}+\frac{4}{3}\right)\times\left(\frac{5}{14}+\frac{3}{2}\times\frac{3}{7}+\frac{3}{2}\right)$

d) $\frac{x}{y^2}\times\frac{xy-yz}{x}$

e) $\frac{x^2-y^2}{x+y}\div\frac{x-y}{x^3+y^3}$

f) $\left(\frac{4}{x}-\frac{3}{y}\right)\div\left(\frac{5}{x}+\frac{6}{y}\right)$

ACTVITIES I.5-Factorize each of the following expressions, using the laws on squares and cubes as necessary:

a) x^4-x^2

b) x^6-y^6

c) x^3y-xy^3

d) $x^5y^2+x^2y^5$

ACTVITIES I.6-Simplify each of the following expressions, showing all the steps of your argument carefully:

a) $\frac{3}{x(x+2)}+\frac{1}{x^2-2x}-\frac{2}{x^2-4}$

b) $\frac{1}{x^2+xy}+\frac{1}{y^2+xy}$

c) $\frac{x}{x-y}-\frac{y}{x+y}-\frac{2xy}{x^2-y^2}$

d) $\left(x+\frac{y^2}{x-y}\right)\div\frac{x^3+y^3}{x^2-y^2}$

e) $x^3-y^3+x^2y-x$

CHAPTER II: LINEAR EQUATIONS AND GRAPHS

II.1: INTRODUCTION

In this Chapter we will discuss some algebraic methods for solving equations and inequalities. Then we will introduce coordinate systems that allow us to explore the relationship between algebra and geometry. Finally, we will use this algebraic–geometric relationship to find equations that can be used to describe real-world data sets. Thus we will learn how to find the equation of a line that fits data and consider many applied problems that can be solved using the concepts discussed in this chapter.

II.2: LINEAR EQUATIONS AND INEQUALITIES

The equation:

$$3 - 2(x + 3) = \frac{x}{3} - 5$$

and the inequality:

$$\frac{x}{2} + 2(3x - 1) \geq 5$$

are both first degree in one variable. In general, a **first-degree**, or **linear equation** in one variable is any equation that can be written in **this standard form**: $ax+b=0,\ a\neq 0$ **(1)**

If the equality symbol, =, in (1) is replaced by $<$; $>$; $\leq$; $\geq$ the resulting expression is called a **first-degree**, or **linear inequality**. A **solution** of **an equation (or inequality)** involving **a single variable** is a number that **when substituted for the variable makes the equation (or inequality) true**. The set of all solutions is called the **solution set**. When we say that we **solve an equation** (or inequality), we mean that we find its solution set. Knowing what is meant by the solution set is one thing; finding it is another. We start by recalling the idea of equivalent equations and equivalent inequalities.

If we perform an operation on an equation (or inequality) that produces another equation (or inequality) with the same solution set, then the two equations (or inequalities) are said to be **equivalent**. The basic idea in solving equations or inequalities is to perform operations that produce simpler equivalent equations or inequalities and to continue the process until we obtain an equation or inequality with an obvious solution.

II.2.1: LINEAR EQUATIONS

Linear equations are generally solved using the following equality properties:

An equivalent equation will result if:

1. The same quantity is added to or subtracted from each side of a given equation.
2. Each side of a given equation is multiplied by or divided by the same nonzero quantity.

Example 1: Solve and check: $8x – 3(x – 4) = 3(x – 4) + 6$

Solution: $8x – 3(x – 4) = 3(x – 4) + 6$ **Use the distributive property.**

$8x - 3x + 12 = 3x - 12 + 6$ **Combine like terms.**

$5x + 12 = 3x – 6$ **Subtract 3*x* from both sides.**

$2x + 12 = -6$ **Subtract 12 from both sides.**

$2x = -18$ **Divide both sides by 2.**

$x = -9$

CHECK:

$$\begin{aligned} 8x - 3(x - 4) &= 3(x - 4) + 6 \\ 8(-9) - 3[(-9) - 4] &\stackrel{?}{=} 3[(-9) - 4] + 6 \\ -72 - 3(-13) &\stackrel{?}{=} 3(-13) + 6 \\ -33 &\stackrel{\checkmark}{=} -33 \end{aligned}$$

EXAMPLE 2:

Solve and check: $\dfrac{x + 2}{2} - \dfrac{x}{3} = 5$

Solution:

What operations can we perform on

$$\frac{x + 2}{2} - \frac{x}{3} = 5$$

to eliminate the denominators? If we can find a number that is exactly divisible by each denominator, we can use the multiplication property of equality to clear the denominators.The LCD (least common denominator) of the fractions, 6, is exactly what we are looking for! Actually, any common denominator will do, but the LCD results in a simpler equivalent equation. So, we multiply both sides of the equation by 6:

$$6\left(\frac{x + 2}{2} - \frac{x}{3}\right) = 6 \cdot 5$$

$$\overset{3}{\cancel{6}} \cdot \frac{(x + 2)}{\underset{1}{\cancel{2}}} - \overset{2}{\cancel{6}} \cdot \frac{x}{\underset{1}{\cancel{3}}} = 30$$

$$\begin{aligned} 3(x + 2) - 2x &= 30 && \text{Use the distributive property.} \\ 3x + 6 - 2x &= 30 && \text{Combine like terms.} \\ x + 6 &= 30 && \text{Subtract 6 from both sides.} \\ x &= 24 \end{aligned}$$

CHECK:

$$\begin{aligned} \frac{x + 2}{2} - \frac{x}{3} &= 5 \\ \frac{24 + 2}{2} - \frac{24}{3} &\stackrel{?}{=} 5 \\ 13 - 8 &\stackrel{?}{=} 5 \\ 5 &\stackrel{\checkmark}{=} 5 \end{aligned}$$

Matched Problem 1:

Solve and check:

$$\frac{x + 1}{3} - \frac{x}{4} = \frac{1}{2}$$

In many applications of algebra, **formulas** or **equations** must be changed to alternative equivalent forms. The following example is typical.

EXAMPLE 3: Solving a Formula for a Particular Variable

If you deposit a **principal *P*** in an account that earns **simple interest at an annual rate *r***, then *A* in the account after *t* years is given by: $A = P + Prt$.

Solve for

(a) *r* in terms of *A*, *P*, and *t*

(b) *P* in terms of *A*, *r*, and *t*

SOLUTION (a):

$$A = P + Prt \quad \text{Reverse equation.}$$
$$P + Prt = A \quad \text{Subtract } P \text{ from both sides.}$$
$$Prt = A - P \quad \text{Divide both members by } Pt.$$
$$r = \frac{A - P}{Pt}$$

(b):

$$A = P + Prt \quad \text{Reverse equation.}$$
$$P + Prt = A \quad \text{Factor out } P \text{ (note the use of the distributive property).}$$
$$P(1 + rt) = A \quad \text{Divide by } (1 + rt).$$
$$P = \frac{A}{1 + rt}$$

Matched Problem 2:

If a cardboard box has length *L*, width *W*, and height *H*,then its surface area is given by the formula $S = 2LW + 2LH + 2WH$. Solve the formula for (a) ***L*** in terms of *S*, *W*, and *H* (a) ***H*** in terms of *S*, *L*, and *W*.

II.2.2: LINEAR INEQUALITIES

Before we start solving linear inequalities, let us recall what we mean by < (less than) and >(greater than). If *a* and *b* are real numbers, we write

$$a < b \quad a \text{ is less than } b$$

if there exists a positive number *p* such that $a + p = b$. Certainly, we would expect that if a positive number was added to any real number, the sum would be larger than the original. That is essentially what the definition states. If $a<b$, we may also write

$$b > a \quad b \text{ is greater than } a.$$

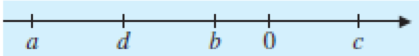
Example 4:

(a) $3 < 5$ **Since 3 + 2 = 5**

(b) $-6 < -2$ **Since -6 + 4 = -2**

(c) $0 > -10$ **Since -10< 0 (because -10 + 10 = 0)**

The inequality symbols have a very clear geometric interpretation on the real number line. If $a < b$, then a is to the left of b on the number line; if $c > d$, then c is to the right of d on the number line:



An equivalent inequality will result, and the **sense or direction will remain the same** if each side of the original inequality:

1. has the same real number added to or subtracted from it.
2. is multiplied or divided by the same *positive* number.

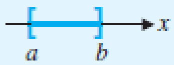
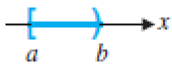
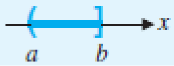
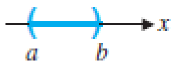
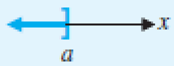
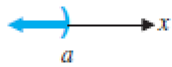
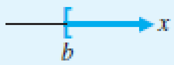
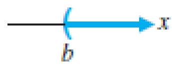
An equivalent inequality will result, and the **sense or direction will reverse** if each side of the original inequality

3. is multiplied or divided by the same *negative* number.
Note: Multiplication by 0 and division by 0 are not permitted.

Therefore, we can perform essentially the same operations on inequalities that we perform on equations, with the exception that **the sense of the inequality reverses if we multiply or divide both sides by a negative number**. Otherwise, the sense of the inequality does not change. For example, if we start with the true statement $-3 > -7$ and multiply both sides by 2, we obtain $-6 > -14$ and the sense of the inequality stays the same. But if we multiply both sides of $-3 > -7$ by -2, the left side becomes 6 and the right side becomes 14, so we must write $6 < 14$ to have a true statement. The **sense** of the inequality **reverses.**

If $a < b$, the **double inequality** $a < x < b$ means that $a < x$ **and** $x < b$; that is, x is between a and b. **Interval notation** is also used to describe sets defined by inequalities, as shown in **Table 1**:

Table 1: **Interval Notation**

Interval Notation	Inequality Notation	Line Graph
$[a, b]$	$a \le x \le b$	
$[a, b)$	$a \le x < b$	
$(a, b]$	$a < x \le b$	
(a, b)	$a < x < b$	
$(-\infty, a]$	$x \le a$	
$(-\infty, a)$	$x < a$	
$[b, \infty)$	$x \ge b$	
(b, ∞)	$x > b$	

The numbers *a* and *b* **in table 1** are called the **endpoints** of the interval. An interval is **closed** if it contains all its endpoints and **open** if it does not contain any of its endpoints. The intervals [*a*, *b*],$(-\infty, a]$ and $[b, \infty)$ are closed, and the intervals (*a*, *b*),$(-\infty, a]$ and (b, ∞) are open.

Note that the symbol ∞ (read infinity) is not a number. When we write $[b, \infty)$, we are simply referring to the interval that starts at *b* and continues indefinitely to the right. We never refer to as ∞ an endpoint, and we never write $[b, \infty]$.The interval $(-\infty, \infty)$ is the entire real number line. Note that an endpoint of a line graph in table 1 has a square bracket through it if the endpoint is included in the interval; a parenthesis through an endpoint indicates that it is not included.

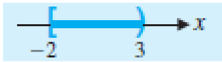
Example 5:

(a) Write [-2, 3) as a double inequality and graph.

(b) Write $x \geq -5$ in interval notation and graph.

SOLUTION- (a):

$[-2, 3)$ is equivalent to $-2 \leq x < 3$.



-(b):

$x \geq -5$ is equivalent to $[-5, \infty)$.

Explore and Discuss:

The solution to Example 5b shows the graph of the inequality $x \geq -5$. What is the graph of $x < -5$? What is the corresponding interval? Describe the relationship between these sets.

Example 6:

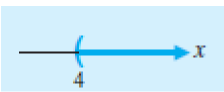
- Solve and graph:

$$2(2x + 3) < 6(x - 2) + 10$$

Solution:

$2(2x + 3) < 6(x - 2) + 10$	Remove parentheses.
$4x + 6 < 6x - 12 + 10$	Combine like terms.
$4x + 6 < 6x - 2$	Subtract 6*x* from both sides.
$-2x + 6 < -2$	Subtract 6 from both sides.
$-2x < -8$	Divide both sides by −2 and reverse the sense of the inequality.

$$x > 4 \quad \text{or} \quad (4, \infty)$$



Matched Problem:

Solve and graph: $-8 \leq 3x - 5 < 7$

Note that a linear equation usually has exactly one solution, while a linear inequality usually has infinitely many solutions.

Applications

To realize the full potential of algebra, we must be able to translate real-world problems into mathematics. In short, we must be able to do word problems. Here are some suggestions that will help you get started:

Procedure for Solving Word Problems

1. Read the problem carefully and introduce a variable to represent an unknown quantity in the problem. Often the question asked in a problem will indicate the unknown quantity that should be represented by a variable.

2. Identify other quantities in the problem (known or unknown), and whenever possible, express unknown quantities in terms of the variable you introduced in **Step 1.**

3. Write a verbal statement using the conditions stated in the problem and then write an equivalent mathematical statement (equation or inequality).

4. Solve the equation or inequality and answer the questions posed in the problem.

5. Check the solution(s) in the original problem.

Example 7 Purchase Price

Alex purchases a plasma TV, pays 7% state sales tax, and is charged $65 for delivery. If Alex's total cost is $1,668.93, what was the purchase price of the TV?

Solution

Step 1

Introduce a variable for the unknown quantity. After reading the problem,
we decide to let x represent the purchase price of the TV.

Step 2

Identify quantities in the problem.

Delivery charge: $65

Sales tax: $0.07x$

Total cost: $1,668.93

Step 3

Write a verbal statement and an equation.

Price + Delivery Charge + Sales Tax = Total Cost

$x + 65 + 0.07x = 1{,}668.93$

Step 4

Solve the equation and answer the question.

$x + 65 + 0.07x = 1{,}668.93$ **Combine like terms**.

$1.07x + 65 = 1{,}668.93$ Subtract 65 from both sides.

$1.07x = 1{,}603.93$ **Divide both sides by 1.07.**

$x = 1{,}499$

The price of the TV is $1,499.

Step 5

Check the answer in the original problem.

Price =	$ 1,499.0 0
Delivery charge =	$ 65.00
Tax = 0.07 x 1,499 =	$ 104.93
Total =	**$ 1,668.93**

Matched Problem:

Mary paid 8.5% sales tax and a $190 title and license fee when she bought a new car for a total of $28,400. What is the purchase price of the car?

The next example involves the important concept of **break-even analysis**, which is encountered in several places in this text. Any manufacturing company has **costs**, and **revenues**, ***R***. The company will have a **loss** if $R < C$, will **break even** if $R = C$, and will have a **profit** if $R > C$. Costs involve **fixed costs**, such as plant overhead, product design, setup, and promotion, and **variable costs**, which are dependent on the number of items produced at a certain cost per item.

Example 8: Break-Even Analysis

A multimedia company produces DVDs.One time fixed costs for a particular DVD are $ 48,000, which include costs such as filming, editing, and promotion. Variable costs amount to $12.40 per DVD and include manufacturing, packaging, and distribution costs for each DVD actually sold to a retailer.
The DVD is sold to retail outlets at $17.40 each. How many DVDs must be manufactured?
and sold in order for the company to break even?

Solution

Step 1
Let x = number of DVDs manufactured and sold.
Step 2
C = cost of producing x DVDs
R = **revenue** (return) on sales of x DVDs
Fixed costs = $ 48,000
Variable costs = 12.40x$
C = Fixed costs + variable costs
= $ 48,000 + $ 12.40x
R = $ 17.40x
Step 3
The company breaks even if $R = C$; that is, if 17.40x$ = $48,000 + 12.40x$
Step 4
$17.4x = 48,000 + 12.4x$ *Subtract* 12.4x from both sides.
$5x = 48,000$ Divide both sides by 5.
$\boldsymbol{x = 9,600}$

The company must make and sell 9,600 DVDs to break even.

Step 5

Check:

Costs	**Revenue**
48,000 + 12.4x (9,600= **$ 167,040)**	17.4x (9,600) = **$ 167,04**

II.3: GRAPHS AND LINES

In this section, we will consider **one of the most basic geometric figures**—a line. When we use the term *line,* we mean ***straight line***. We will learn how to recognize and graph a line, and how to use information concerning a line to find its equation. Examining the graph of any equation often results in additional insight into the nature of the equation's solutions.

II.3.1: CARTESIAN COORDINATE SYSTEM

Recall that to form a **Cartesian** or **rectangular coordinate system**, we select two real number lines—one horizontal and one vertical—and let them cross through their origins as indicated in **figure 1**. Up and to the right are the usual choices for the positive directions. These two number lines are called the **horizontal axis** and the **vertical axis**, or, together, the **coordinate axes**. The horizontal axis is usually referred to as the ***x*** **axis** and the vertical axis as the ***y*** **axis**, and each is labeled accordingly. The coordinate axes divide the plane into four parts called **quadrants**, which are numbered counterclockwise from **I** to **IV** (**see fig. 1**).

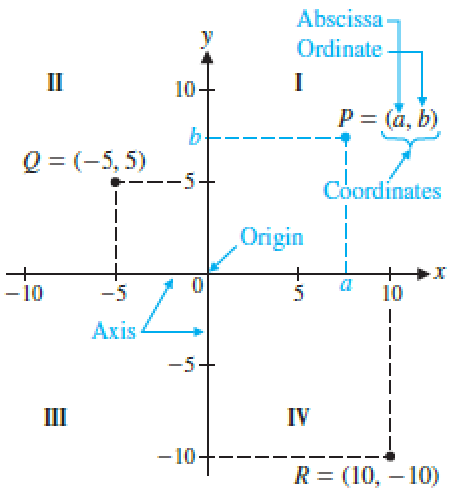


FIGURE 1: THE CARTESIAN (RECTANGULAR) COORDINATE SYSTEM

Now we want to assign *coordinates* to each point in the plane. Given an arbitrary point *P* in the plane, pass horizontal and vertical lines through the point (**fig**. 1). The vertical line will intersect the horizontal axis at a point with coordinate *a*, and the horizontal line will intersect the vertical axis at a point with coordinate *b*. These two numbers, written as the **ordered pair** (*a*, *b*) form the **coordinates** of the point *P*. The first coordinate, *a*, is called the **abscissa** of *P*; the second coordinate, *b*, is called the **ordinate** of *P*. The abscissa of Q **in figure** 1 is -5, and the ordinate of Q is 5. The coordinates of a point can also be referenced in terms of the axis labels. The ***x*** **coordinate** of *R* in figure 1 is 10, and the ***y*** **coordinate** of *R* is -10. The point with coordinates (0, 0)is called the **origin**. The procedure we have just described assigns to each point *P* in the plane a unique pair of real numbers (*a*, *b*). Conversely, if we are given an ordered pair of real numbers (*a*, *b*), then, reversing this procedure, we can determine a unique point *P* in the plane. Thus,

There is a one-to-one correspondence between the points in a plane and the elements in the set of all ordered pairs of real numbers. This is often referred to as the **fundamental theorem of analytic geometry**.

II.3.2: GRAPHS OF $Ax + By = C$

In previous section, we called an equation of the form $ax + b = 0$ ($a \neq 0$) a linear equation in one variable. Now we want to consider linear equations in **two** variables. A **linear equation in two variables** is an equation that can be written in the **standard form:** $Ax + By = C$ where *A*, *B*, and *C* are
variables.

A **solution** of an equation in two variables is an ordered pair of real numbers that satisfies the equation. For example,(4, 3) is a solution of $3x - 2y = 6$. The **solution set** of an equation in two variables is the set of all solutions of the equation. The **graph** of an equation is the graph of its solution set.

Explore and Discuss:

(a) As noted earlier, (4, 3) is a solution of the equation.

$$3x - 2y = 6$$

Find three more solutions of this equation. Plot these solutions in a Cartesian Coordinate system. What familiar geometric shape could be used to describe the solution set of this equation?

(**b**) Repeat part (A) for the equation $x = 2$.

(**c**) Repeat part (A) for the equation $y = -3$.

In explore and discuss,you may have recognized that the graph of each equation is a (straight) line.

The graph of any equation of the form $Ax + By = C$ (*A* and *B* not both 0) (1) is **a line**, and **any line in a Cartesian coordinate system** is **the graph of an equation of this form**.

If $A \neq 0$ and $B \neq 0$, then equation (**1**) can be written as

$$y = -\frac{A}{B}x + \frac{C}{B} = mx + b,$$

If $A = 0$ and $B \neq 0$, then equation (1) can be written as

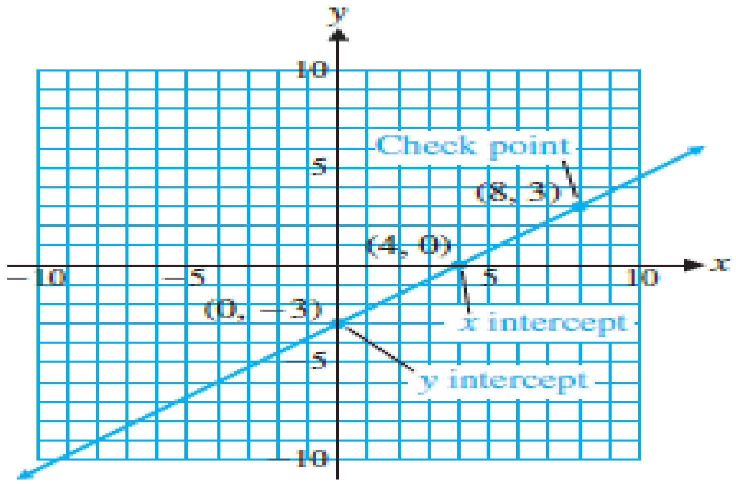
$Y = \frac{C}{B}$ and its graph is a **horizontal line**. If $A \neq 0$ and $B = 0$, then equation (1) can be Written as $x = \frac{C}{A}$ and its graph is a **vertical line**. To graph equation (1), or any of its special cases, plot any two points in the solution set and use a straightedge to draw the line through these two points. The points where the line crosses the axes are often the easiest to find. The *y* intercept* is the *y* coordinate of the point where the graph crosses the *y* axis, and the *x* intercept is the *x* coordinate of the point where the graph crosses the *x* axis. To find the *y* intercept, let $x = 0$ and solve for *y*. To find the *x* intercept, let $y = 0$ and solve for *x*. It is a good idea to find a third point as a check point..

Example 1: Using Intercepts to Graph a Line from that **equation**: $3x - 4y = 12$

SOLUTION

x	y	
0	−3	y intercept
4	0	x intercept
8	3	Check point

The graph is:



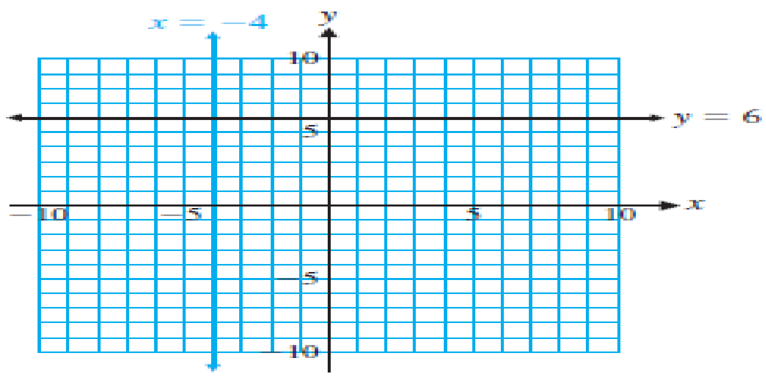
Matched Problem- Graph: $4x - 3y = 12$

Example 2: Horizontal and Vertical Lines

(**a**) Graph $x = -4$ and $y = 6$ simultaneously in the same rectangular coordinate system.
(**b**) Write the equations of the vertical and horizontal lines that pass through the point (7, -5).

Solution

(**a**)



(b) **Horizontal line** through (7, -5): $y = -5$; Vertical **line** through (7, -5): $x = 7$

Matched Problem:
(a) Graph $x = 5$ and $y = -3$ simultaneously in the same rectangular coordinate system.
(b) Write the equations of the vertical and horizontal lines that pass through the point (-8, 2).

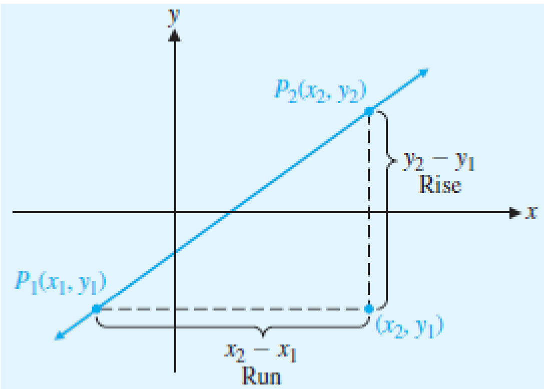
II.3.2.1: SLOPE OF A LINE

If we take two points, P_1 (x_1, y_1) and P_2(x_2, y_2), on a line, then the ratio of the change in *y* to the change in *x* as the point moves from point P_1 to point P_2 is called the **slope** of the line. In a sense, slope provides a measure of the "steepness" of a line relative to the *x* axis. The change in *x* is often called the **run**, and the change in *y* is the **rise**.

DEFINITION-SLOPE OF A LINE.

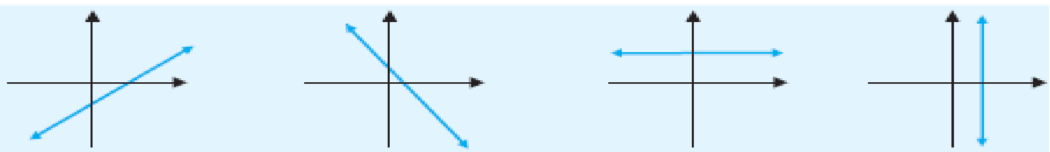
If a line passes through two distinct points, $\boldsymbol{P_1}$ **($\boldsymbol{x_1}$, $\boldsymbol{y_1}$)** and $\boldsymbol{P_2}$**($\boldsymbol{x_2}$, $\boldsymbol{y_2}$)** , then its slope is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_1 \neq x_2$$
$$= \frac{\text{vertical change (rise)}}{\text{horizontal change (run)}}$$

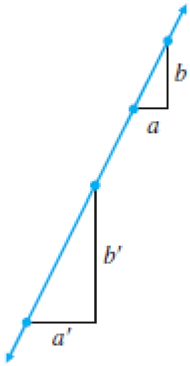


For a horizontal line, *y* does not change; its slope is 0. For a vertical line, *x* does not change; $x_1 = x_2$ so its slope is not defined. In general, the slope of a line may be positive, negative, 0, or not defined. Each case is illustrated geometrically in table **1 below:**

TABLE 1: GEOMETRIC INTERPRETATION OF SLOPE

Line	Rising as x moves from left to right	Falling as x moves from left to right	Horizontal	Vertical
Slope	Positive	Negative	0	Not defined
Example				

The slope of a line is the same for any pair of distinct points on the line as shown below:



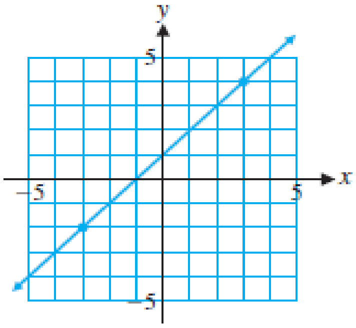
$$m = \frac{b}{a} = \frac{b'}{a'}$$

Example 3: **Finding Slopes**

*Sketch a line through each pair of points, and find the slope of each line:
(**a**) (-3, -2); (3, 4) (**b**) (-1, 3); (2, -3) (**c**) (-2, -3); (3, -3) (**d**) (-2, 4), (-2, -2)

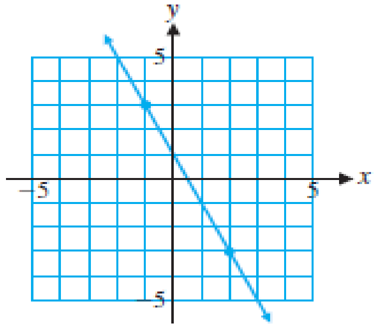
Solution:

a)



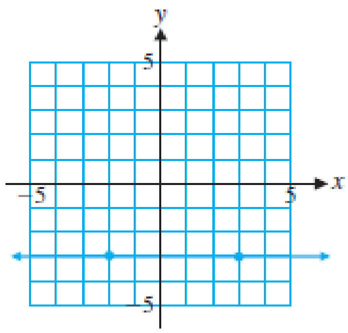
$$m = \frac{4 - (-2)}{3 - (-3)} = \frac{6}{6} = 1$$

b)



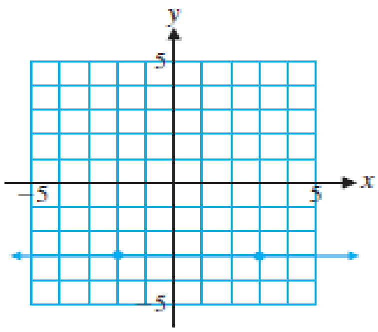
$$m = \frac{-3 - 3}{2 - (-1)} = \frac{-6}{3} = -2$$

c)



$$m = \frac{-3 - (-3)}{3 - (-2)} = \frac{0}{5} = 0$$

d)



$$m = \frac{-3 - (-3)}{3 - (-2)} = \frac{0}{5} = 0$$

Slopes not defined

Matched Problem:

*Find the slope of the line through each pair of points.

(**a**) (-2, 4); (3, 4)   (**b**) (-2, 4);(0, -4)   (**c**) (-1, 5);(-1, -2)   (**d**) (-1, -2); (2, 1)

II.3.2.2: EQUATIONS OF LINES: SPECIAL FORMS

Let us start by investigating why $y = mx + b$ is called the ***slope-intercept form*** for a line.

Explore and Discuss:

*(a) Graph $y = x + b$ for b = -5, -3, 0, 3, and 5 simultaneously in the same coordinate system. Verbally describe the geometric significance of b.

*(b) Graph $y = mx - 1$ for m = -2, -1, 0, 1, and 2 simultaneously in the same coordinate system. Verbally describe the geometric significance of m.

As you may have deduced from **Explore and Discuss**, constants ***m*** and b in $y = mx + b$ have the following geometric interpretations:

If we let $x = 0$, then $y = b$. So the graph of $y = mx + b$ crosses the y axis at $(0, b)$. **The constant *b* is the *y intercept***. For example, the ***y* intercept** of the graph of $y = -4x - 1$ is **-1**. To determine the geometric significance of m, we proceed as follows: If $y = mx + b$, then by setting $x = 0$ and $x = 1$, we conclude that $(0, b)$ and $(1, m + b)$ lie on its graph (**Figure 2**).

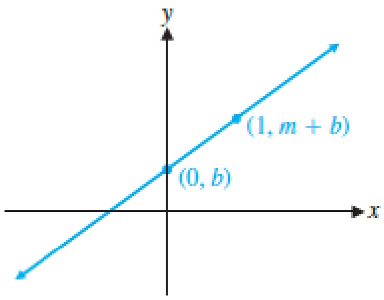


Figure 2: The ***slope-intercept form*** for a line.

The slope of this line is given by:

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(m + b) - b}{1 - 0} = m$$

So ***m*** is the slope of the line given by $y = mx + b$.

Thus **the equation** $y = mx + b$ (m = slope, $b = y$ intercept) is called the slope-intercept form of an equation of a line.

Example 4: Using the Slope-Intercept Form

(a) Find the slope and y intercept, and graph $y = -2/3\, x - 3$.

(b) Write the equation of the line with slope 2/3 and y intercept -2.

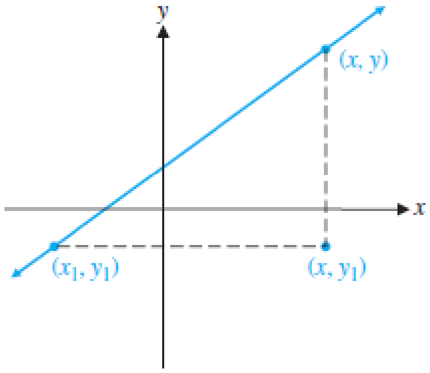
Solution

(a) Slope = m = -2/3 ; y intercept = b = -3

(b) m = 2/3 and b = -2; so, y = 2/3x-2

Matched Problem: Write the equation of the line with slope 1/2and y intercept-1. Grap**h.**

Suppose that a line has slope *m* and passes through a fixed point (x_1, y_1). If the point (*x*, *y*) is any other point on the line **(Fig. 3**),



then $\frac{y-y1}{x-x1} = m$. That is, $y - y_1 = m(x - x_1)$ -**point-slope form of an equation of a line**.

We now observe that (x_1, y_1) also satisfies equation the previous equation and conclude that **it is** an **equation of a line** with slope *m* that passes through (x_1, y_1). **The point-slope form** is extremely useful, since it enables us to find an equation for a line if we know its slope and the coordinates of a point on the line or if we know the coordinates of two points on the line.

Example 5: Using the Point-Slope Form

(**a**) Find an equation for the line that has slope 1/2 and passes through (-4, 3). Write the final answer in the form Ax + *By* = *C*.

(**b**) Find an equation for the line that passes through the points (-3, 2) and (-4, 5). Write the resulting equation in the form *y* = *mx* + *b*.

Solution

(**a**)

Use $y - y_1 = m(x - x_1)$. Let $m = \frac{1}{2}$ and $(x_1, y_1) = (-4, 3)$. Then

$$y - 3 = \tfrac{1}{2}[x - (-4)]$$

$$y - 3 = \tfrac{1}{2}(x + 4) \qquad \text{Multiply both sides by 2.}$$

$$2y - 6 = x + 4$$

$$-x + 2y = 10 \quad \text{or} \quad x - 2y = -10$$

b)

First, find the slope of the line by using the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{-4 - (-3)} = \frac{3}{-1} = -3$$

Now use $y - y_1 = m(x - x_1)$ with $m = -3$ and $(x_1, y_1) = (-3, 2)$:

$$y - 2 = -3[x - (-3)]$$

$$y - 2 = -3(x + 3)$$

$$y - 2 = -3x - 9$$

$$y = -3x - 7$$

Matched Problem:

(**a**) Find an equation for the line that has slope 2/3 and passes through (6, -2).Write the resulting equation in **the form *Ax* + *By* = *C*, *A* $>$0**.
(**b**) Find an equation for the line that passes through (2, -3) and (4, 3). Write the resulting equation in **the form *y* = *mx* + *b***.

II.3.2.3: APPLICATIONS

We will now see how equations of lines occur in certain applications.

Example 6: Cost Equation

The management of a company that manufactures skateboards has fixed costs (costs at 0 output) of $300 per day and total costs of $4,300 per day at an output of 100 skateboards per day. Assume that cost *C* is linearly related to output *x*.
(**a**) Find the slope of the line joining the points associated with outputs of 0 and 100; that is, the line passing through(0, 300) and (100, 4,300).
(**b**) Find an equation of the line relating output to cost. Write the final answer in the form *C* = *mx* + *b*.
(**c**)Graph the cost equation from part (**b**) for $0 \le x \le 200$.

Solution

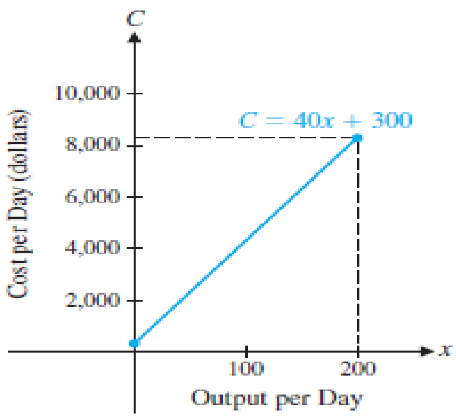
a)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4{,}300 - 300}{100 - 0} = \frac{4{,}000}{100} = 40$$

b) We must find an equation of the line that passes through 10, 3002 with slope 40. We use the slope-intercept form:

$$C = mx + b$$
$$C = 40x + 300$$

c)



The *fixed cost* of \$300 per day covers plant cost, insurance, and so on. This cost is incurred whether or not there is any production. The *variable cost* is $40x$, which depends on the day's output. Since increasing production from x to $x + 1$ will increase the cost by \$40 (from $40x + 300$ to $40x + 340$), the slope 40 can be
interpreted as the **rate of change** of the cost function with respect to production x.

Matched Problem: Answer parts (**a**) and (**b**) in **Example 6** for fixed costs of \$250 per day and total costs of \$3,450 per day at an output of 80 skateboards per day.

In a free competitive market, the price of a product is determined by **the relationship between supply and demand**. If there is a surplus—that is, the supply is greater than the demand—the price tends to come down. If there is a shortage—that is, the demand is greater than the supply—the price tends to go up. The price tends to move toward an equilibrium price at which the supply and demand are equal. The following example introduces **the basic concepts**.

Example 7: Supply and Demand

At a price of \$9.00 per box of oranges, the supply is 320,000 boxes and the demand is 200,000 boxes. At a price of \$8.50 per box, the supply is 270,000 boxes and the demand is 300,000 boxes.
(**a**) Find a price–supply equation of the form $p = mx + b$, where p is the price in dollars and x is the corresponding supply in thousands of boxes.
(**b**) Find a price–demand equation of the form $p = mx + b$, where p is the price in dollars and x is the corresponding demand in thousands of boxes.
(**c**) Graph the price–supply and price–demand equations in the same coordinate system and find their point of intersection.

Solution

a) To find a price–supply equation of the form $p = mx + b$, we must find two points of the form (x, p) that are on the supply line. From the given supply data, (320, 9) and (270, 8.5) are two such points. First, find the slope of the line:

$$m = \frac{9 - 8.5}{320 - 270} = \frac{0.5}{50} = 0.01$$

Now use **the point-slope form** to find the equation of the line:

$$\begin{aligned} p - p_1 &= m(x - x_1) && (x_1, p_1) = (320, 9) \\ p - 9 &= 0.01(x - 320) \\ p - 9 &= 0.01x - 3.2 \\ p &= 0.01x + 5.8 && \text{Price–supply equation} \end{aligned}$$

b) From the given demand data, (200, 9) and (300, 8.5) are two points on the demand line.

$$m = \frac{8.5 - 9}{300 - 200} = \frac{-0.5}{100} = -0.005$$

$$p - p_1 = m(x - x_1) \qquad (x_1, p_1) = (200, 9)$$

$$p - 9 = -0.005(x - 200)$$

$$p - 9 = -0.005x + 1$$

$$p = -0.005x + 10 \qquad \text{Price–demand equation}$$

c)From part (a), we plot the points (320, 9) and (270, 8.5) and then draw the line through them. We do the same with the points (200, 9) and (300, 8.5) from part (b) (**Fig. 4**). (Note that we restricted the axes to intervals that contain these data points.) To find the intersection point of the two lines, we equate the right hand sides of the price–supply and price–demand equations and solve for x:

Price-supply Price-demand

$$0.01x + 5.8 = -0.005x + 10$$

$$0.015x = 4.2$$

$$x = 280$$

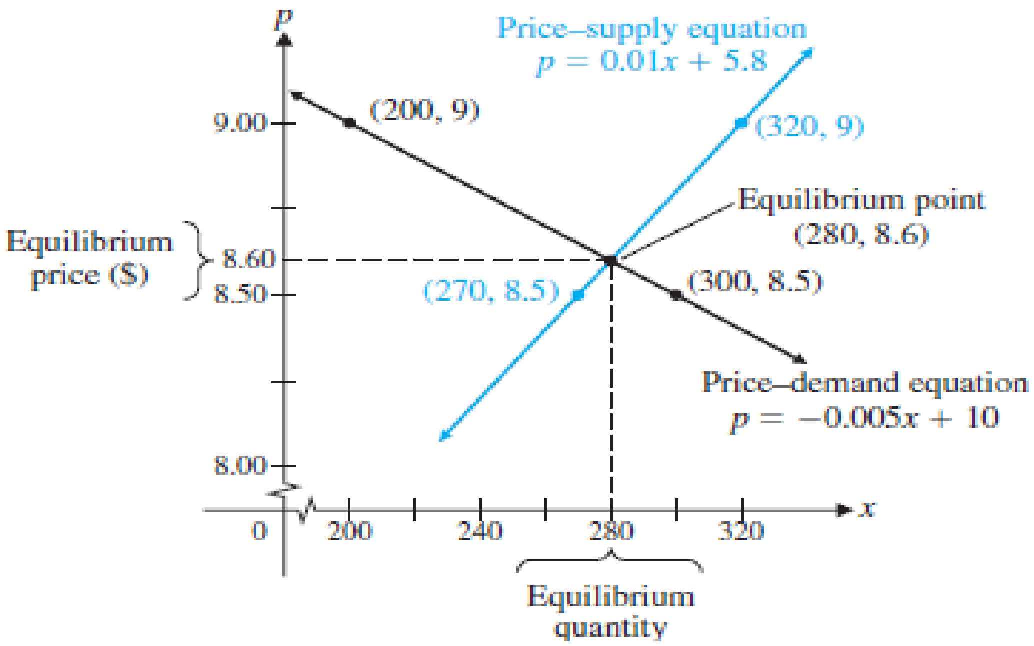


Figure 4: Graphs of price–supply and price–demand equations

Now use the price–supply equation to find p when $x = 280$:

$$p = 0.01x + 5.8$$

$$p = 0.01(280) + 5.8 = 8.6$$

As a check, we use the price–demand equation to find p when $x = 280$:

$$p = -0.005x + 10$$

$$p = -0.005(280) + 10 = 8.6$$

The lines intersect at (280, 8.6). The intersection point of the price–supply and price–demand equations is called the **equilibrium point**, and its coordinates are the **equilibrium quantity** (280) and the **equilibrium price** ($8.60). These terms are illustrated in **Figure 4.**

CHAPTER III: INEQUALITIES AND ABSOLUTE VALUES

III.1: SOME SIMPLE INEQUALITIES

Basic inequalities concerning the real numbers are simple, provided that we exercise due care. We begin by studying the effect of addition and multiplication on inequalities.

ADDITION AND MULTIPLICATION RULES.

Suppose that $a,b \in$ R and $a < b$. Then

(a) for every $c \in$ R, we have $a + c < b + c$;

(b) for every $c \in$ R satisfying $c > 0$, we have $ac < bc$; and (c) for every $c \in$ R satisfying $c < 0$, we have $ac > bc$.In other words, addition by a real number c preserves the inequality. On the other hand, multiplication by a real number c preserves the inequality if $c > 0$ and reverses the inequality if $c < 0$.

Remark. We can deduce some special rules for positive real numbers. Suppose that $a,b,c,d \in$ R are all positive. If $a < b$ and $c < d$, then $ac < bd$. To see this, note simply that by part (b) above, we have $ac < bc$ and $bc < bd$.

SQUARE AND RECIPROCAL RULES.

Suppose that $a,b \in$ R and $0 < a < b$. Then

(a) $a^2 < b^2$; and (b) $a^{-1} > b^{-1}$.

Proof. Part (a) is a special case of **our Remark** if we take $c = a$ and $d = b$. To show part (b), note that

$$a^{-1} - b^{-1} = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} > 0.$$

CAUCHY'S INEQUALITY.

For every $a,b \in$ R, we have $a^2 + b^2 \geq 2ab$. Furthermore, equality holds precisely when $a = b$.

Proof.Simply note that $a^2 + b^2 - 2ab = a^2 - 2ab + b^2 = (a - b)^2 \geq 0$, and that equality holds precisely when $a - b = 0$. We now use some of the above rules to solve inequalities. We shall illustrate the ideas by considering a few examples in some detail.

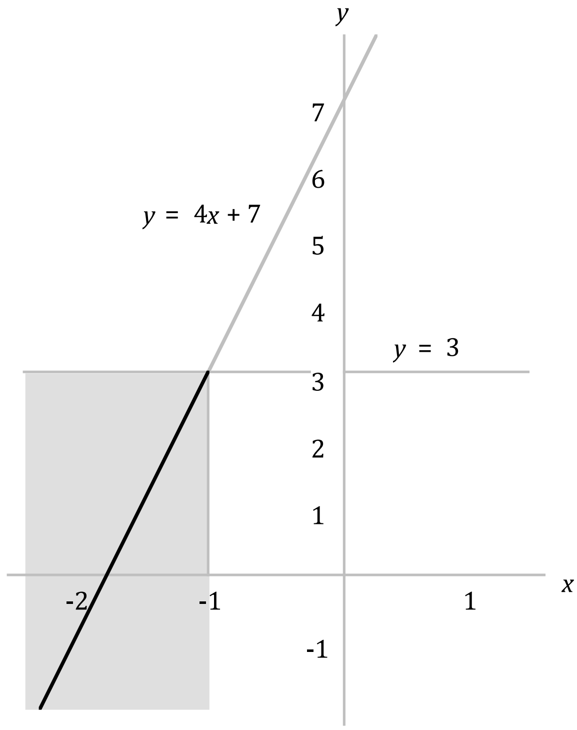
Example 1: **Consider the inequality $4x + 7 < 3$**. Using the Addition rule and adding −7 to both sides, we obtain $4x < -4$. Using one of the Multiplication rules and multiplying both sides by the positive real number 1/4, we obtain $x < -1$. We have shown that

$$4x + 7 < 3 \quad \Longrightarrow \quad x < -1.$$

Suppose now that $x < -1$. Using one of the Multiplication rules and multiplying both sides by the positive real number 4, we obtain $4x < -4$. Using the Addition rule and adding 7 to both sides, we obtain $4x + 7 < 3$. Combining this with our earlier observation, we have now shown that

$$4x + 7 < 3 \iff x < -1.$$

We can confirm our conclusion by drawing a graph of the line $y = 4x+7$ and observing that the part of the line below the horizontal line $y = 3$ corresponds to $x < -1$ on the x-axis.



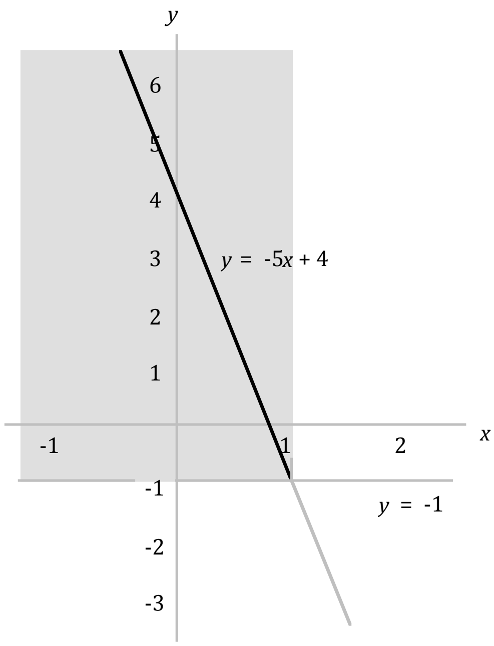
Example 2: **Consider the inequality $-5x + 4 > -1$**. Using one of the Multiplication rules and multiplying both sides by the negative real number −1, we obtain $5x - 4 < 1$. Using the Addition rule and adding 4 to both sides, we obtain $5x < 5$. Using one of the Multiplication rules and multiplying both sides by the positive real number 1⁄5, we obtain $x < 1$. We have shown that

$$-5x + 4 > -1 \implies x < 1.$$

Suppose now that $x < 1$. Using one of the Multiplication rules and multiplying both sides by the positive real number 5, we obtain $5x < 5$. Using the Addition rule and adding −4 to both sides, we obtain $5x-4 < 1$. Using one of the Multiplication rules and multiplying both sides by the negative real number −1, we obtain $-5x + 4 > -1$. Combining this with our earlier observation, we have now shown that

$$-5x + 4 > -1 \iff x < 1.$$

We can confirm our conclusion by drawing a graph of the line $y = -5x + 4$ and observing that the part of the line above the horizontal line $y = -1$ corresponds to $x < 1$ on the x-axis.



Example 3: **Consider the inequality $x^2 \leq a^2$**, where $a > 0$ is fixed. Clearly $x = \pm a$ are the only solutions of the equation $x^2 = a^2$. So let us consider the inequality $x^2 < a^2$. Observe first of all that the inequality is satisfied when $x = 0$. On the other hand, if $0 < x < a$, then the Square rule gives $x^2 < a^2$. However, if $-a < x < 0$, then using one of the Multiplication rules and multiplying all sides by the negative real number -1, we obtain $a > -x > 0$. It follows from the Square rule that $(-x)^2 < a^2$, so that $x^2 < a^2$. We have now shown that

$$-a \leq x \leq a \quad \Longrightarrow \quad x^2 \leq a^2.$$

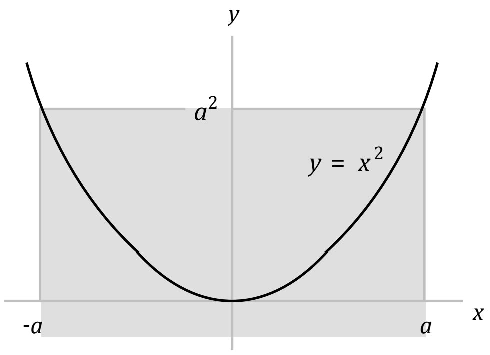
Suppose now that $x > a$. Then it follows from the Square rule that $x^2 > a^2$. On the other hand, suppose that $x < -a$. Using one of the Multiplication rules and multiplying both sides by the negative real number -1, we obtain $-x > a$. It follows from the Square rule that $(-x)^2 > a^2$, so that $x^2 > a^2$. We have now shown that

$$x < -a \quad \text{or} \quad x > a \quad \Longrightarrow \quad x^2 > a^2.$$

It now follows that

$$-a \leq x \leq a \quad \Longleftrightarrow \quad x^2 \leq a^2.$$

We can confirm our conclusion by drawing a graph of the parabola $y = x^2$ and observing that the part of the parabola on or below the horizontal line $y = a^2$ corresponds to $-a \leq x \leq a$ on the x-axis.



Example 4: Consider the inequality $x^2 - 4x + 3 \leq 0$. We can write

$$x^2 - 4x + 3 = x^2 - 4x + 4 - 1 = (x - 2)^2 - 1,$$

so that the inequality is equivalent to $(x - 2)^2 - 1 \leq 0$, which in turn is equivalent to the inequality $(x - 2)^2 \leq 1$, in view of the Addition rule. Now write $u = x - 2$. Then it follows from Example 6.1.3 that

$$-1 \leq u \leq 1 \quad \Longleftrightarrow \quad u^2 \leq 1.$$

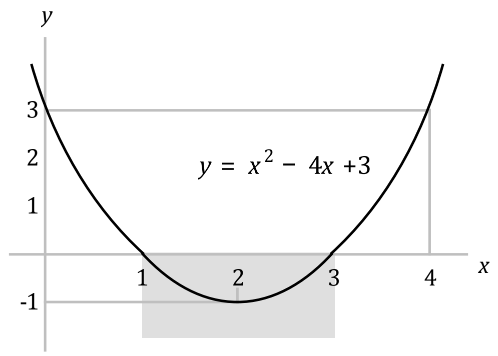
Hence

$$-1 \leq x - 2 \leq 1 \quad \Longleftrightarrow \quad (x - 2)^2 \leq 1.$$

Using the addition rule on the inequalities on the left hand side, and using our earlier observation, we conclude that

$$1 \leq x \leq 3 \quad \Longleftrightarrow \quad x^2 - 4x + 3 \leq 0.$$

We can confirm our conclusion by drawing a graph of the parabola $y = x^2 - 4x + 3$ and observing that the part of the parabola on or below the horizontal line $y = 0$ corresponds to $1 \leq x \leq 3$ on the x-axis.



Example 5: Consider the inequality below.

$$\frac{1}{x} < 2.$$

Clearly we cannot have $x = 0$, as $1/0$ is meaningless. We have two cases:

1)-Suppose that $x > 0$. Using one of the Multiplication rules and multiplying both sides by the positive real number x, we obtain the inequality $1 < 2x$. Multiplying both sides by the positive real number $1/2$, we obtain $1/2 < x$. Suppose now that $1/2 < x$. Using one of the Multiplication rules and multiplying both sides by the positive real number $2/x$, we obtain the original inequality. We have therefore shown that for $x > 0$, we have

$$x > \frac{1}{2} \quad \Longleftrightarrow \quad \frac{1}{x} < 2.$$

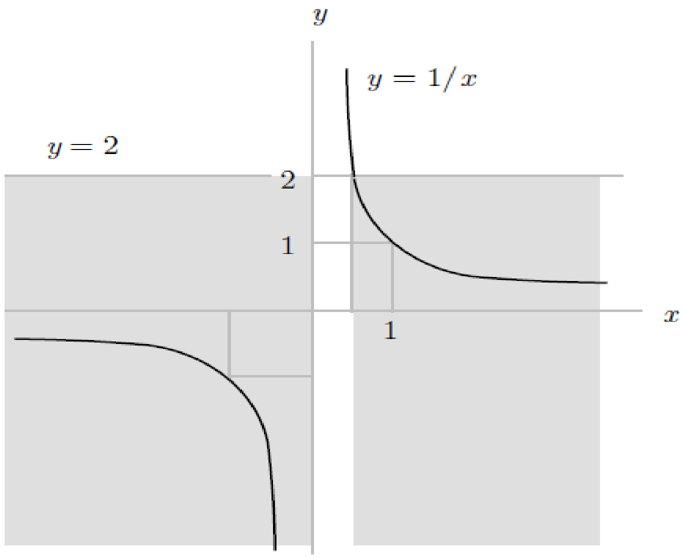
2)-Try to use one of the Multiplication rules to show that

$$x < 0 \quad \Longleftrightarrow \quad \frac{1}{x} < 0.$$

The result is obvious, but the proof is slightly tricky. Combining the two parts, we conclude that

$$x < 0 \quad \text{or} \quad x > \frac{1}{2} \quad \Longleftrightarrow \quad \frac{1}{x} < 2.$$

We can confirm our conclusion by drawing a graph of the hyperbola $y = 1/x$ and observing that the part of the hyperbola below the horizontal line $y = 2$ corresponds to $x < 0$ together with $x > 1/2$ on the x-axis.



III.2: ABSOLUTE VALUES

Definition. For every $a \in$ R, the absolute value $|a|$ of a is a non-negative real number satisfying

$$|a| = \begin{cases} a & \text{if } a \geq 0; \\ -a & \text{if } a < 0. \end{cases}$$

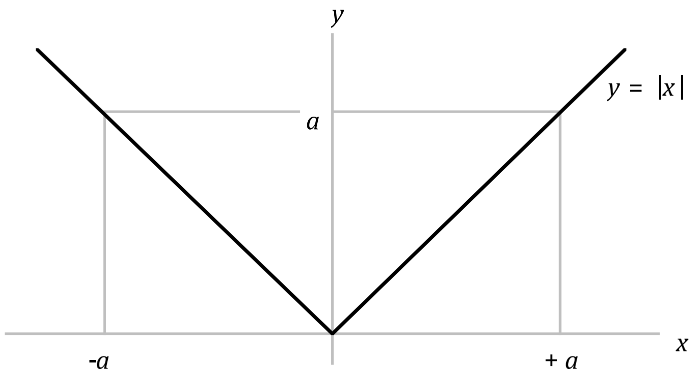
Remark. If we place the number ***a*** on the real number line, then the absolute value $|a|$ represents the distance of a from the origin 0.

PROPERTIES OF ABSOLUTE VALUES.

For every $a, b \in$ R, we have

(a) $|a| \geq 0$;
(b) $|a| \geq a$;
(c) $|a|^2 = a^2$;
(d) $|ab| = |a||b|$; and
(e) $|a + b| \leq |a| + |b|$.

The graph of the function $y = |x|$ is given below, where $a \in R$ is a non-negative real number.



As $a \in R$ is non negative, then

$|x| \leq a$ if and only if $-a \leq x \leq +a$, and $|x| < a$ if and only if $-a < x < +a$.

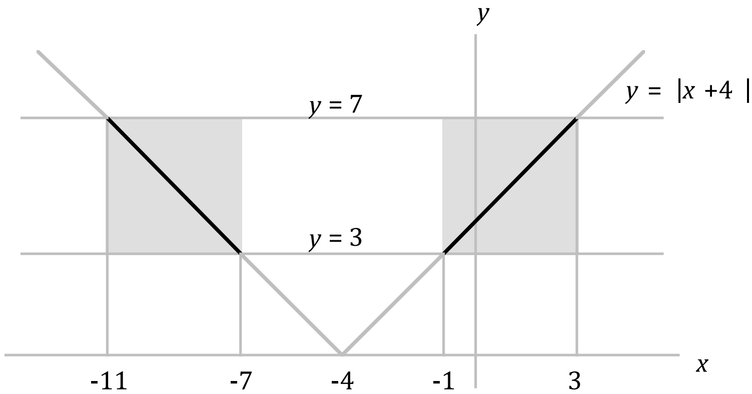
Example 6: The equation $|x| = 4$ has two solutions $x = \pm 4$.
Example 7. The equation $|2x + 1| = 5$ has two solutions, one satisfying $2x + 1 = 5$ and the other satisfying $2x + 1 = -5$. Hence $x = 2$ or $x = -3$.

Example 8: The inequality $|x| < 5$ is satisfied precisely when $-5 < x < 5$.

Example 9:. The inequality $|2x + 1| \leq 9$ is satisfied precisely when $-9 \leq 2x + 1 \leq 9$; in other words, when $-5 \leq x \leq 4$.

Example 10: The equation $\sqrt{x^2 + 4x + 13} = x - 1$ is satisfied only if the right hand side is nonnegative, so that we must have $x \geq 1$. Squaring both sides, we have $x^2+4x+13 = (x-1)^2 = x^2-2x+1$, so that $6x + 12 = 0$, giving $x = -2$. Hence the equation has no real solution x.

Example 11. Consider the inequalities $3 < |x+4| < 7$. Note first of all that the inequality $|x+4| < 7$ holds precisely when $-7 < x + 4 < 7$; in other words, when $-11 < x < 3$. On the other hand, the inequality $|x+4| \leq 3$ holds precisely when $-3 \leq x+4 \leq 3$; in other words, precisely when $-7 \leq x \leq -1$. Hence the inequality $|x + 4| > 3$ holds precisely when $x < -7$ or $x > -1$. It follows that the original inequalities hold precisely when $-11 < x < -7$ or $-1 < x < 3$. We can confirm our conclusion by drawing a graph of the function $y = |x + 4|$ and observing that the part of the graph between the horizontal lines $y = 3$ and $y = 7$ corresponds to $-11 < x < -7$ together with $-1 < x < 3$ on the x-axis.


y
y = |x +4 |
y = 7
y = 3
-11
-7
-4
-1
3
x

ACTIVITIES OF CHAPTER III

ACTVITIES III.1: Suppose that *α* and *β* are two positive real numbers. The number $\frac{1}{2}(\alpha+\beta)$ is called the arithmetic mean of *α* and *β*, while the number $\sqrt{\alpha\beta}$ is called the geometric mean of *α* and *β*.

a) Prove that $\sqrt{\alpha\beta} \le \frac{1}{2}(\alpha+\beta)$; in other words, the geometric mean never exceeds the arithmetic mean.
b) Show that equality holds in part (a) precisely when $\alpha = \beta$.

ACTVITIES III.2: For each of the following inequalities, find all real values of *x* satisfying the inequality:

a) $2x + 4 < 6$ b) $5 - 3x > 11$ c) $7x + 9 > -5$ d) $4x + 4 < 28$

e) $2x + 5 < 3$ f) $4 - 6x \ge 10$

ACTVITIES III.3: Determine all real values of *x* for which the inequalities $5 < 2x + 7 \le 13$ hold.
ACTVITIES III.4: For each of the following inequalities, determine all real values of *x* for which the inequality holds, taking care to distinguish the two cases $x > 0$ and $x < 0$, and explain each step of your argument by quoting the relevant rules concerning inequalities:

a) $\dfrac{x+4}{2x} < 3$ b) $\dfrac{1}{x} < 3$ c) $-2 < \dfrac{1}{x} < 3$

ACTVITIES III.5: For each of the following inequalities, determine all real values of *x* for which the inequality holds, taking care to distinguish two cases, and explain each step of your argument by quoting the relevant rules concerning inequalities:

a) $\dfrac{2x+3}{3x+1} < 1$ b) $\dfrac{4x-2}{x+4} \ge 2$ c) $2 \le \dfrac{4x-2}{x+4} < 3$

1. Find all solutions of the inequality picture. $|x + 2| < 6$, and confirm your answer by drawing a suitable

2. For each of the following inequalities, determine all real values of *x* for which the inequality holds:
   a) $1 < |3x - 5| \le 7$ b) $1 \le |(x - 1)^3| \le 8$

CHAPTER IV: SYSTEMS OF LINEAR EQUATIONS

IV.1: INTRODUCTION

Most linear systems of any consequence involve large numbers of equations and variables. It is impractical to try to solve such systems by hand. There are a wide array of approaches to solving linear systems, ranging from graphing calculators to software and spreadsheets. In this chapter,we--------

IV.2: SYSTEMS OF LINEAR EQUATIONS

Let us consider a pair of simultaneous linear equations (**system A**) in two variables, of the type

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in R$. Multiplying the first equation in (**A**) by b_2 and multiplying the second equation in (**A**) by b_1, we obtain (**B**):

$$a_1b_2x + b_1b_2y = c_1b_2,$$
$$a_2b_1x + b_1b_2y = c_2b_1.$$

Subtracting the second equation in (**B**) from the first equation, we obtain

$$(a_1b_2x + b_1b_2y) - (a_2b_1x + b_1b_2y) = c_1b_2 - c_2b_1.$$

Some simple algebra leads to(**C**):

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1.$$

On the other hand, multiplying the first equation in (**A**) by a_2 and multiplying the second equation in (A) by a_1, we obtain (D):

$$a_1a_2x + b_1a_2y = c_1a_2,$$
$$a_1a_2x + b_2a_1y = c_2a_1.$$

Subtracting the second equation in (**D**) from the first equation, we obtain

$$(a_1a_2x + b_1a_2y) - (a_1a_2x + b_2a_1y) = c_1a_2 - c_2a_1.$$

Some simple algebra leads to **(E)**

$$(b_1a_2 - b_2a_1)y = c_1a_2 - c_2a_1.$$

Suppose that:

$$a_1b_2 - a_2b_1 \neq 0.$$

Then (**C**) and (**E**) can be written in the form **F**)

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{c_1a_2 - c_2a_1}{b_1a_2 - b_2a_1}.$$

In practice, we do not need to remember these formulas. It is much easier to do the calculations by using some common sense and cutting a few corners in doing so.
We have the following geometric interpretation. Each of the two linear equations in (**A**) represents a line on the *xy*-plane. The condition

$$a_1 b_2 - a_2 b_1 \neq 0.$$

ensures that the two lines are not parallel, so that they intersect at precisely one point, given by (**F**)

Example1:. Suppose that

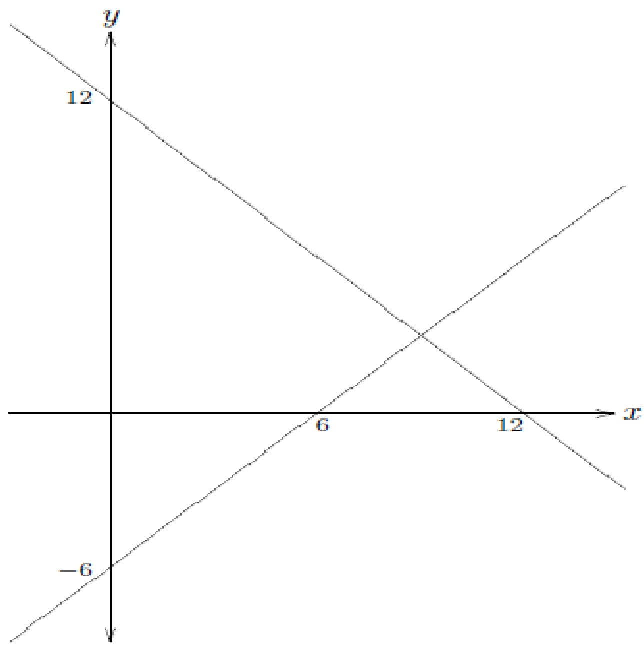
$$x + y = 12,$$

$$x - y = 6.$$

Note that we can eliminate *y* by adding the two equations. More precisely, we have

$$(x + y) + (x - y) = 12 + 6.$$

This gives $2x = 18$, so that $x = 9$. We now substitute the information $x = 9$ into one of the two original equations. Simple algebra leads to $y = 3$. We have the following picture.



Example 2:. Suppose that

$$x + y = 32,$$

$$3x + 2y = 70.$$

We can multiply the first equation by 2 and keep the second equation as it is to obtain

$$2x + 2y = 64, 3x + 2y = 70.$$

The effect of this is that both equations have a term $2y$. We now subtract the first equation from the second equation to eliminate this common term. More precisely, we have

$$(3x + 2y) - (2x + 2y) = 70 - 64.$$

This gives $x = 6$. We now substitute the information $x = 6$ into one of the two original equations. Simple algebra leads to $y = 26$.

Example 3. Suppose that

$$3x + 2y = 10,$$

$$4x - 3y = 2.$$

We can multiply the first equation by 4 and the second equation by 3 to obtain

$$12x + 8y = 40,$$

$$12x - 9y = 6.$$

The effect of this is that both equations have a term $12x$. We now subtract the second equation from the first equation to eliminate this common term. More precisely, we have

$$(12x + 8y) - (12x - 9y) = 40 - 6.$$

This gives $17y = 34$, so that $y = 2$. We now substitute the information $y = 2$ into one of the two original equations. Simple algebra leads to $x = 2$. The reader may try to eliminate the variable y first and show that we must have $x = 2$.

Example 4:. Suppose that

$$7x - 5y = 16$$
$$2x + 7y = 13.,$$

We can multiply the first equation by 7 and the second equation by 5 to obtain

$$49x - 35y = 112,$$
$$10x + 35y = 65.$$

The effect of this is that both equations have a term $35y$ but with opposite signs. We now add the two equation to eliminate this common term. More precisely, we have

$$(49x - 35y) + (10x + 35y) = 112 + 65.$$

This gives $59x = 177$, so that $x = 3$. We now substitute the information $x = 3$ into one of the two original equations. Simple algebra leads to $y = 1$.

Example 5: Suppose that the difference between two numbers is equal to 11, and that twice the smaller number minus 4 is equal to the larger number. To find the two numbers, let x denote the larger number and y denote the smaller number. Then we have $x - y = 11$ and $2y - 4 = x$, so that

$$x - y = 11,$$

$$x - 2y = -4.$$

We now eliminate the variable x by subtracting the second equation from the first equation. More precisely, we have

$$(x - y) - (x - 2y) = 11 - (-4).$$

This gives $y = 15$. We now substitute the information $y = 15$ into one of the two original equations. Simple algebra leads to $x = 26$.

Example 6:. Suppose that a rectangle is 5cm longer than it is wide. Suppose also that if the length and width are both increased by 2cm, then the area of the rectangle increases by 50cm^2. To find the dimension of the rectangle, let x denote its length and y denote its width. Then we have $x = y+5$ and $(x+2)(y+2)-xy = 50$. Simple algebra shows that the second equation is the same as $2x+2y+4 = 50$. We therefore have

$$\begin{aligned} x - \quad y &= \quad 5 \\ 2x + 2y &= 46., \end{aligned}$$

We can multiply the first equation by 2 and keep the second equation as it is to obtain

$$\begin{aligned} 2x - 2y &= 10 \\ 2x + 2y &= 46., \end{aligned}$$

We now eliminate the variable y by adding the two equations. More precisely, we have

$$(2x - 2y) + (2x + 2y) = 10 + 46.$$

This gives $4x = 56$, so that $x = 14$. It follows that $y = 9$.

The idea of eliminating one of the variables can be extended to solve systems **of three linear equations.** We illustrate the ideas by the following examples**.**

Example 6: Suppose that

$$x + y + z = 6,$$

$$2x + 3y + z = 13,$$

$$x + 2y - z = 5.$$

Adding the first equation and the third equation, or adding the second equation and the third equation, we eliminate the variable z on both occasions and obtain respectively

$$2x + 3y = 11, \; 3x + 5y = 18.$$

Solving this system, the reader can show that $x = 1$ and $y = 3$. Substituting back to one of the original equations, we obtain $z = 2$.

Example 7. Suppose that

$$x - y + z = 10, \; 4x + 2y - 3z = 8, \; 3x - 5y + 2z = 34.$$

We can multiply the three equations by 6, 2 and 3 respectively to obtain

$$6x - 6y + 6z = 60, \qquad 8x + 4y - 6z = 16, \qquad 9x - 15y + 6z = 102.$$

The reason for the multiplication is to arrange for the term $6z$ to appear in each equation to make the elimination of the variable z easier. Indeed, adding the first equation and the second equation, or adding the second equation and the third equation, we eliminate the variable z on both occasions and obtain respectively:

$$14x - 2y = 76,\ 17x - 11y = 118.$$

Multiplying the first equation by 11 and the second equation by 2, we obtain

$$154x - 22y = 836,\ 34x - 22y = 236.$$

Eliminating the variable y, we obtain $120x = 600$, so that $x = 5$. It follows that $y = -3$. Using now one of the original equations, we obtain $z = 2$.

Example 8: Suppose that

$$6x + 4y - 2z = 0,$$

$$3x - 2y + 4z = 3,\ 5x - 2y + 6z = 3.$$

Multiplying the last two equations by 2, we obtain

$$6x + 4y - 2z = 0,$$

$$6x - 4y + 8z = 6,\ 10x - 4y + 12z = 6.$$

The reason for the multiplication is to arrange for the term $4y$ to appear in each equation to make the elimination of the variable y easier. Indeed, adding the first equation and the second equation, or adding the first equation and the third equation, we eliminate the variable y on both occasions and obtain respectively

$$12x + 6z = 6,\ 16x + 10z = 6.$$

Solving this system, the reader can show that $x = 1$ and $z = -1$. Substituting back to one of the original equations, we obtain $y = -2$.

Example 9: Suppose that

$$2x + y - z = 9,\ 5x + 2z = -3,\ 7x - 2y = 1.$$

Our strategy here is to eliminate the variable y between the first and third equations. To do this, the first equation can be written in the form $4x+2y - 2z = 18$. Adding this to the third equation, and also keeping the second equation as it is, we obtain

$$11x - 2z = 19,\ 5x + 2z = -3.$$

Solving this system, the reader can show that $x = 1$ and $z = -4$. Substituting back to one of the original equations, we obtain $y = 3$. The reader may also wish to first eliminate the variable z between the first two equations and obtain a system of two equations in x and y.

IV.3: QUADRATIC EQUATIONS

Consider an equation of the type

$$ax^2 + bx + c = 0, \qquad \text{(A)}$$

where $a,b,c \in$R are constants and $a \neq 0$ .To solve such an equation, we observe first of all that

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0$$

precisely when

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \qquad \textbf{(B)}$$

There are three cases:

(1) If $b^2 - 4ac < 0$, then the right hand side of (**B)** is negative. It follows that (**B**) is never satisfied for any real number x, so that the equation (**A**) has no real solution.

(2) If $b^2 - 4ac = 0$, then (**B**) becomes

$$\left(x + \frac{b}{2a}\right)^2 = 0, \qquad \text{so that} \qquad x = -\frac{b}{2a}.$$

Indeed, this solution occurs twice, as we shall see later.

(3) If $b^2 - 4ac > 0$, then (**B**) becomes

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \qquad \text{so that} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two distinct real solutions for the equation (A).

Example 1. For the equation $2x^2+6x+4 = 0$, we have $(a,b,c) = (2,6,4)$, so that $b^2-4ac = 4 > 0$. It follows that this equation has two distinct real solutions, given by

$$x = \frac{-6 \pm \sqrt{4}}{4} = -1 \text{ or } -2.$$

Observe that $2x^2 + 6x + 4 = 2(x + 1)(x + 2)$.

Example 2: For the equation $x^2+2x+3 = 0$, we have $(a,b,c) = (1,2,3)$, so that $b^2-4ac = -8 < 0$. It follows that this equation has no solution.

Example 3. For the equation $3x^2 - 12x + 12 = 0$, we have $b^2 - 4ac = 0$. It follows that this equation has one real solution, given by $x = 2$. Observe that $3x^2 - 12x + 12 = 3(x - 2)^2$. This is the reason we say that the root occurs twice.

IV.3: FACTORIZATION AGAIN

Consider equation (**A**) again. Sometimes we may be able to find a factorization of the form

$$ax^2 + bx + c = a(x - \alpha)(x - \beta), \qquad \text{(C)}$$

where $\alpha, \beta \in R$. Clearly $x = \alpha$ and $x = \beta$ are solutions of the equation (9).

Example 1: For the equation $x^2 - 5x = 0$, we have the factorization

$$x^2 - 5x = x(x - 5) = (x - 0)(x - 5).$$

It follows that the two solutions of the equation are $x = 0$ and $x = 5$.

Example 2:. For the equation $x^2 - 9 = 0$, we have the factorization $x^2 - 9 = (x - 3)(x + 3)$. It follows that the two solutions of the equation are $x = \pm 3$.

Note that

$$a(x - \alpha)(x - \beta) = a(x^2 - (\alpha + \beta)x + \alpha\beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta.$$

It follows from (C) that

$$ax^2 + bx + c = ax^2 - a(\alpha + \beta)x + a\alpha\beta.$$

Equating corresponding coefficients, we obtain

$$b = -a(\alpha + \beta) \qquad \text{and} \qquad c = a\alpha\beta.$$

We have proved the following result-**SUM AND PRODUCT OF ROOTS OF A QUADRATIC EQUATION.**

Suppose that $x = \alpha$ and $x = \beta$ are the two roots of a quadratic equation $ax^2 + bx + x = 0$. Then

$$\alpha + \beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = \frac{c}{a.}$$

Example 3.For the equation $x^2 - 5x - 7 = 0$, we have $(a,b,c) = (1,-5,-7)$, and

$$x = \frac{5 \pm \sqrt{25 + 28}}{2} = \frac{5 \pm \sqrt{53}}{2}.$$

Note that

$$\frac{5 + \sqrt{53}}{2} + \frac{5 - \sqrt{53}}{2} = 5 \qquad \text{and} \qquad \frac{5 + \sqrt{53}}{2} \times \frac{5 - \sqrt{53}}{2} = \frac{25 - 53}{4} = -7.$$

Example 4. For the equation $x^2 - 13x + 4 = 0$, we have $(a,b,c) = (1,-13,4)$, and

$$x = \frac{13 \pm \sqrt{169-16}}{2} = \frac{13 \pm \sqrt{153}}{2}.$$

Note that

$$\frac{13+\sqrt{153}}{2} + \frac{13-\sqrt{153}}{2} = 13 \quad \text{and} \quad \frac{13+\sqrt{153}}{2} \times \frac{13-\sqrt{153}}{2} = \frac{169-153}{4} = 4.$$

We conclude this section by studying a few more examples involving factorization of quadratic polynomials.

Example 5. Consider the expression $x^2 - 4x + 3$. The roots of the equation $x^2 - 4x + 3 = 0$ are given by

$$\alpha = \frac{4+\sqrt{16-12}}{2} = 3 \quad \text{and} \quad \beta = \frac{4-\sqrt{16-12}}{2} = 1.$$

Hence we have $x^2 - 4x + 3 = (x-3)(x-1)$.

Example 6. Consider the expression $2x^2 + 5x + 2$. The roots of the equation $2x^2 + 5x + 2 = 0$ are given by

$$\alpha = \frac{-5+\sqrt{25-16}}{4} = -\frac{1}{2} \quad \text{and} \quad \beta = \frac{-5-\sqrt{25-16}}{4} = -2.$$

Hence we have

$$2x^2 + 5x + 2 = 2\left(x + \frac{1}{2}\right)(x+2) = (2x+1)(x+2).$$

Example 7. Consider the expression $4x^2 - x - 14$. The roots of the equation $4x^2 - x - 14 = 0$ are given by

$$\alpha = \frac{1+\sqrt{1+224}}{8} = 2 \quad \text{and} \quad \beta = \frac{1-\sqrt{1+224}}{8} = -\frac{7}{4}.$$

Hence we have

$$4x^2 - x - 14 = 4(x-2)\left(x + \frac{7}{4}\right) = (x-2)(4x+7).$$

Example.8 We have

$(x + 2)^2 - (2x - 1)^2 = (x^2 + 4x + 4) - (4x^2 - 4x + 1) = x^2 + 4x + 4 - 4x^2 + 4x - 1 = -3x^2 + 8x + 3$. The roots

of the equation $-3x^2 + 8x + 3 = 0$ are given by

$$\alpha = \frac{-8+\sqrt{64+36}}{-6} = -\frac{1}{3} \quad \text{and} \quad \beta = \frac{-8-\sqrt{64+36}}{-6} = 3.$$

Hence

$$(x+2)^2 - (2x-1)^2 = -3\left(x + \frac{1}{3}\right)(x-3) = -(3x+1)(x-3) = (3x+1)(3-x).$$

Alternatively, we can use one of the Laws on squares (writing $a = x + 2$ and $b = 2x - 1$). We have

$(x + 2)^2 - (2x - 1)^2 = ((x + 2) - (2x - 1))((x + 2) + (2x - 1))$

$= (x + 2 - 2x + 1)(x + 2 + 2x - 1) = (3 - x)(3x + 1).$

Example 9: Consider the expression $6p - 17pq + 12pq^2$. Taking out a factor p, we have $6p - 17pq + 12pq^2 = p(6 - 17q + 12q^2)$.

Consider next the quadratic factor $6 - 17q + 12q^2$. The roots of the equation $6 - 17q + 12q^2 = 0$ are given by

$$\alpha = \frac{17 + \sqrt{289 - 288}}{24} = \frac{3}{4} \quad \text{and} \quad \beta = \frac{17 - \sqrt{289 - 288}}{24} = \frac{2}{3}.$$

Hence

$$6p - 17pq + 12pq^2 = p(6 - 17q + 12q^2) = 12p\left(q - \frac{3}{4}\right)\left(q - \frac{2}{3}\right) = p(4q - 3)(3q - 2).$$

Example 10. We have $10a^2b + 11ab - 6b = b(10a^2 + 11a - 6)$. Consider the quadratic factor $10a^2 + 11a - 6$. The roots of the equation $10a^2 + 11a - 6 = 0$ are given by

$$\alpha = \frac{-11 + \sqrt{121 + 240}}{20} = \frac{2}{5} \quad \text{and} \quad \beta = \frac{-11 - \sqrt{121 + 240}}{20} = -\frac{3}{2}.$$

Hence

$$10a^2b + 11ab - 6b = b(10a^2 + 11a - 6) = 10b\left(a - \frac{2}{5}\right)\left(a + \frac{3}{2}\right) = b(5a - 2)(2a + 3).$$

Example 11: Consider the expression

$$\frac{a - 3}{a^2 - 11a + 28} - \frac{a + 4}{a^2 - 6a - 7}.$$

Note that $a^2 - 11a + 28 = (a - 7)(a - 4)$ and $a^2 - 6a - 7 = (a - 7)(a + 1)$. Hence

$$\begin{aligned}\frac{a - 3}{a^2 - 11a + 28} - \frac{a + 4}{a^2 - 6a - 7} &= \frac{a - 3}{(a - 7)(a - 4)} - \frac{a + 4}{(a - 7)(a + 1)} \\ &= \frac{(a - 3)(a + 1)}{(a - 7)(a - 4)(a + 1)} - \frac{(a + 4)(a - 4)}{(a - 7)(a - 4)(a + 1)} \\ &= \frac{(a - 3)(a + 1) - (a + 4)(a - 4)}{(a - 7)(a - 4)(a + 1)} = \frac{(a^2 - 2a - 3) - (a^2 - 16)}{(a - 7)(a - 4)(a + 1)} \\ &= \frac{13 - 2a}{(a - 7)(a - 4)(a + 1)}.\end{aligned}$$

Example 12.: We have

$$
\begin{aligned}
&\frac{1}{x+1}+\frac{1}{(x+1)(x+2)}-\frac{4}{(x+1)(x+2)(x+3)} \\
&\qquad=\frac{(x+2)(x+3)}{(x+1)(x+2)(x+3)}+\frac{(x+3)}{(x+1)(x+2)(x+3)}-\frac{4}{(x+1)(x+2)(x+3)} \\
&\qquad=\frac{(x+2)(x+3)+(x+3)-4}{(x+1)(x+2)(x+3)}=\frac{(x^2+5x+6)+(x+3)-4}{(x+1)(x+2)(x+3)}=\frac{x^2+6x+5}{(x+1)(x+2)(x+3)} \\
&\qquad=\frac{(x+1)(x+5)}{(x+1)(x+2)(x+3)}=\frac{x+5}{(x+2)(x+3)}.
\end{aligned}
$$

IV.4: HIGHER ORDER EQUATIONS

For polynomial equations of degree greater than 2, we do not have general formulae for their solutions. However, we may occasionally be able to find some solutions by inspection. These may help us find other solutions. We shall illustrate the technique here by using three examples.

Example 1. Consider the equation $x^3-4x^2+2x+1 = 0$. It is easy to see that $x = 1$ is a solution of this cubic polynomial equation. It follows that $x - 1$ is a factor of the polynomial $x^3 - 4x^2 + 2x + 1$. Using long division, we have the following:

$$
\begin{array}{r|l}
 & x^2 - 3x - 1 \\
\hline
x - 1 & x^3 - 4x^2 + 2x + 1 \\
 & x^3 - x^2 \\
\hline
 & -3x^2 + 2x \\
 & -3x^2 + 3x \\
\hline
 & -x + 1 \\
 & -x + 1 \\
\hline
\end{array}
$$

Hence $x^3 - 4x^2 + 2x + 1 = (x - 1)(x^2 - 3x - 1)$. The other two roots of the equation are given by the two roots of $x^2 - 3x - 1 = 0$. These are

$$x = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3 \pm \sqrt{13}}{2}.$$

Example 2. Consider the equation $x^3+2x^2-5x-6 = 0$. It is easy to see that $x = -1$ is a solution of this cubic polynomial equation. It follows that $x + 1$ is a factor of the polynomial $x^3 + 2x^2 - 5x - 6$. Using long division, we have the following:

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x + 1 & x^3 + 2x^2 - 5x - 6 \\
 & x^3 + x^2 \\
\hline
 & x^2 - 5x \\
 & x^2 + x \\
\hline
 & -6x - 6 \\
 & -6x - 6 \\
\hline
\end{array}
$$

Hence $x^3 + 2x^2 - 5x - 6 = (x + 1)(x^2 + x - 6)$. The other two roots of the equation are given by the two roots of $x^2 + x - 6 = 0$. These are

$$x = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = 2 \text{ or } -3.$$

Example 3. Consider the equation $x^4 + 7x^3 - 6x^2 - 2x = 0$. It is easy to see that $x = 0$ and $x = 1$ are solutions of this biquadratic polynomial equation. It follows that $x(x - 1)$ is a factor of the polynomial $x^4 +$

$7x^3 - 6x^2 - 2x$. Clearly we have $x^4 + 7x^3 - 6x^2 - 2x = x(x^3 + 7x^2 - 6x - 2)$. On the other hand, using long division, we have the following:

$$
\begin{array}{r|rrrr}
 & x^2+ & 8x+ & 2 & \\
\hline
x-1 & x^3+7x^2- & 6x- & 2 & \\
 & x^3-x^2 & & & \\
\hline
 & 8x^2- & 6x & & \\
 & 8x^2- & 8x & & \\
\hline
 & & 2x- & 2 & \\
 & & 2x- & 2 & \\
\hline
\end{array}
$$

Hence $x^4 + 7x^3 - 6x^2 - 2x = x(x - 1)(x^2 + 8x + 2)$. The other two roots of the equation are given by the two roots of $x^2 + 8x + 2 = 0$. These are

$$x = \frac{-8 \pm \sqrt{64-8}}{2} = \frac{-8 \pm \sqrt{56}}{2} = -4 \pm \sqrt{14}.$$

ACTIVITIES FOR CHAPTER IV

ACTIVITIES IV.1: Solve each of the following equations:

a) $\dfrac{9x-8}{8x-6}=\dfrac{1}{2}$ b) $\dfrac{5-2x}{3x+7}=9$ c) $\dfrac{5x+14}{3x+2}=3$

d) $\dfrac{2x-2}{3}+\dfrac{10-2x}{6}=2x-4$ e) $\dfrac{3x-4}{6x-10}=\dfrac{4x+1}{8x-7}$

ACTIVITIES IV.2: A rectangle is 2 metres longer than it is wide. On the other hand, if each side of the rectangle is increased by 2 metres, then the area increases by 16 square metres. Find the dimension of the rectangle.

ACTIVITIES IV.3: A rectangle is 10 metres wider than it is long. On the other hand, if the width and length are both decreased by 5 metres, then the area of the rectangle decreases by 125 square metres. Find the dimension of the rectangle.

ACTIVITIES IV.4: The lengths of the two perpendicular sides of a right-angled triangle differ by 6 centimetres. On theother hand, if the length of the longer of these two sides is increased by 3 centimetres and the length of the shorter of these two sides is decreased by 2 centimetres, then the area of the right-angled triangle formed is decreased by 5 square centimetres. What is the dimension of the original triangle?

ACTIVITIES IV.5: Solve each of the following systems of linear equations:

a) $3x-2y+4z=11$, $2x+3y+3z=17$, $4x+6y-2z=10$

b) $5x+2y+4z=35$, $2x-3y+2z=19$, $3x+5y+3z=19$

c) $3x+4y+2z=9$, $5x-2y+4z=7$, $2x+6y-2z=6$

d) $x+4y-3z=2$, $x-2y+2z=1$, $x+6y-5z=2$

ACTIVITIES IV.6: Determine the number of solutions of each of the following quadratic equations and find the solutions:

$x^2-84x+98=0$

$x^2-2x-48=0$

a) $3x^2-x+1=0$ b) $4x^2+12x+9=0$ c) $18x^2-84x+98=0$

d) $6x^2-13x+6=0$ e) $5x^2+2x+1=0$ f) $x^2-2x-48=0$

g) $12x^2+12x+3=0$ h) $2x^2-32x+126=0$ i) $3x^2+6x+15=0$

ACTIVITIES IV.7: Factorize each of the following expressions:

a) $14x^2+19x-3$ b) $6x^2+x-12$ c) $(5x+1)^2-20x$

d) $(2x+1)^2+x(2+4x)$ e) $4x^3+9x^2+2x$ f) $x^2y-xy-6y$

g) $8x-2xy-xy^2$

ACTIVITIES IV.8: Simplify each of the following expressions, showing all the steps of your argument carefully:

a) $\dfrac{2}{x-2}+\dfrac{2}{x-5}-\dfrac{x}{x^2-3x+2}-\dfrac{2}{x-1}$ b) $\dfrac{x-3}{x^2-3x+2}-\dfrac{x-2}{x^2-4x+3}$

c) $\dfrac{x^4+x^2y^2+y^4}{x^3-y^3}\div\dfrac{x^2-xy+y^2}{x-y}$ d) $\dfrac{4xy}{(x-y)^2}+\dfrac{x^2-xy}{x^2-y^2}\left(1+\dfrac{y}{x}\right)$

ACTIVITIES IV.8: Study each of the following equations for real solutions:

a) $x^3-6x^2+11x-6=0$ b) $x^3-3x^2+4=0$ c) $x^3+2x^2+6x+5=0$

d) $x^3 - x^2 - x + 1 = 0$ e) $x^3 + 2x^2 - x - 2 = 0$

CHAPTER V: MATRICES

Consider the two linear equations:

$$\begin{aligned} 3x + 4y &= 11 \\ 5x + 7y &= 19., \end{aligned}$$

It is easy to check the two equations are satisfied when $x = 1$ and $y = 2$. We can represent these two linear equations in matrix form as

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \end{pmatrix},$$

where we adopt the convention that

$$\begin{pmatrix} 3 & 4 \\ \bullet & \bullet \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ \bullet \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bullet & \bullet \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bullet \\ 19 \end{pmatrix}$$

represent respectively the information $3x + 4y = 11$ and $5x + 7y = 19$. Under this convention, it is easy to see that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

for every $x,y \in$R. Next, observe that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where, under a convention slightly more general to the one used earlier, we have representing respectively $(7\times3) + ((-4)\times5) = 1$, $(7\times4) + ((-4)\times7) = 0$, $((-5)\times3) + (3\times5) = 0$ and

$((-5) \times 4) + (3 \times 7) = 1$.

. It now follows on the one hand that,

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and on the other hand that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 11 \\ 19 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The convention mentioned in the example above is simply the rule concerning the **multiplication of matrices**. The purpose of this chapter is to study the arithmetic in connection with matrices. We shall be concerned primarily **with 2 × 2 real matrices**. These are arrays of real numbers of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

consisting of **two rows** counted from **top to bottom**, and **two columns** counted from **left to right**. An entry a_{ij} thus corresponds to the entry in row i and column j.

V.1: ARITHMETIC OF MATRICES

***ADDITION AND SUBTRACTION.**

Suppose that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

are two 2 × 2 matrices. Then

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \quad \text{and} \quad A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}.$$

In other words, **we perform addition and subtraction entrywise**. The operations addition and subtraction are governed by the following rules:

(a) Operations within brackets are performed first.
(b) Addition and subtraction are performed in their order of appearance.
(c) A number of additions can be performed in any order. For any 2 × 2 matrices *A,B,C*, we have

$A + (B + C) = (A + B) + C$ and $A + B = B + A$.

Example 1. We have

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right) = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 7 & 13 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 6 \end{pmatrix}.$$

Example 2. We have

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right) = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 7 & 13 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 12 & 20 \end{pmatrix},$$

and

$$\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}\right) + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 9 & 14 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 12 & 20 \end{pmatrix}.$$

Example 3.Like real numbers, it is not true in general that $A - (B - C) = (A - B) - C$. Note that

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right) = \begin{pmatrix} 2 & 2 \\ 4 & 6 \end{pmatrix},$$

and

$$\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}\right) - \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix}$$

Remark. The matrix

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfies $0 + A = A + 0 = A$ for any 2 × 2 matrix *A*, and plays a role analogous to the real number 0 in addition of real numbers. This matrix 0 is called the zero matrix.

***MULTIPLICATION BY A REAL NUMBER**.

Suppose that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a 2 × 2 matrix, and that *r* is a real number. Then

$$rA = \begin{pmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{pmatrix}.$$

In other words, we multiply each entry of *A* by the same real number *r*. This operation is governed by the following rules:

(a) Operations within brackets are performed first.
(b) If there are no brackets to indicate priority, then multiplication by a real number takes precedence over addition and subtraction.
(c) A number of multiplications by real numbers can be performed in any order.

For any 2 × 2 matrix *A* and any real numbers $r,s \in$R, we have $(rs)A = r(sA)$.

(d) For any 2 × 2 matrix *A* and any real numbers $r,s \in$R, we have $(r + s)A = rA + sA$.

(e) For any 2 × 2 matrices *A,B* and any real number $r \in$R, we have $r(A + B) = rA + rB$.

Example 3. We have

$$2\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = 2\begin{pmatrix} 4 & 6 \\ 8 & 11 \end{pmatrix} = \begin{pmatrix} 8 & 12 \\ 16 & 22 \end{pmatrix}.$$

Example 4. We have

$$3\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = 3\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}\right) = 3\begin{pmatrix} 5 & 8 \\ 11 & 15 \end{pmatrix} = \begin{pmatrix} 15 & 24 \\ 33 & 45 \end{pmatrix}.$$

Example 5. We have

$$2\left(3\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\right) = 2\begin{pmatrix} 9 & 12 \\ 15 & 21 \end{pmatrix} = \begin{pmatrix} 18 & 24 \\ 30 & 42 \end{pmatrix} = 6\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = (2 \times 3)\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}.$$

Example 6. We have

$$5\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - 2\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 15 & 20 \\ 25 & 35 \end{pmatrix} - \begin{pmatrix} 6 & 8 \\ 10 & 14 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 15 & 21 \end{pmatrix} = 3\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = (5-2)\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}.$$

***MULTIPLICATION OF MATRICES.**

Suppose that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

are 2 × 2 matrices. Then

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

This operation is governed by the following rules:

(a) Operations within brackets are performed first.

(b) If there are no brackets to indicate priority, then multiplication takes precedence over addition and subtraction.

(c) For any 2 × 2 matrices *A,B,C*, we have (*AB*)*C* = *A*(*BC*).

(d) For any 2 × 2 matrices *A,B* and any real numbers *r* ∈R, we have *r*(*AB*) = (*rA*)*B* = *A*(*rB*).

(e) For any 2 × 2 matrices *A,B,C*, we have *A*(*B* + *C*) = *AB* + *AC* and (*A* + *B*)*C* = *AC* + *BC*.

Remarks 1. Note that the definition above agrees with the convention adopted in **Example II.1.1**. Observe that we have

$$\begin{pmatrix} a_{11} & a_{12} \\ \bullet & \bullet \end{pmatrix} \begin{pmatrix} b_{11} & \bullet \\ b_{21} & \bullet \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & \bullet \\ \bullet & \bullet \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} \\ \bullet & \bullet \end{pmatrix} \begin{pmatrix} \bullet & b_{12} \\ \bullet & b_{22} \end{pmatrix} = \begin{pmatrix} \bullet & a_{11}b_{12} + a_{12}b_{22} \\ \bullet & \bullet \end{pmatrix},$$

$$\begin{pmatrix} \bullet & \bullet \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & \bullet \\ b_{21} & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet \\ a_{21}b_{11} + a_{22}b_{21} & \bullet \end{pmatrix},$$

$$\begin{pmatrix} \bullet & \bullet \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \bullet & b_{12} \\ \bullet & b_{22} \end{pmatrix} = \begin{pmatrix} \bullet & \bullet \\ \bullet & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

2.The matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies *IA* = *AI* = *A* for any 2 × 2 matrix *A*, and plays a role analogous to the real number 1 in multiplication of real numbers. This matrix *I* is called the identity matrix.

3.Multiplication of matrices is generally not commutative; in other words, given two 2×2 matrices *A* and *B*, it is not automatic that *AB* = *BA*. For example, let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix}$$ and
$$BA = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 13 & 20 \\ 5 & 8 \end{pmatrix}.$$

Example 7. We have

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right) \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -2 & 19 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \left(\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -4 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -12 & 19 \end{pmatrix}.$$

,

Example 8. We have

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right) \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -10 & 10 \\ -10 & 10 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 5 \\ -10 & 9 \end{pmatrix} + \begin{pmatrix} -4 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -10 & 10 \\ -10 & 10 \end{pmatrix}.$$

Since I is the identity matrix, we would like to find a technique to obtain, for any given 2×2 matrix A, an inverse 2 × 2 matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. This is not always possible, since in the case of real numbers, the number 0 does not have a multiplicative inverse. We therefore need a condition on 2 × 2 matrices which is equivalent to saying that a real number is non-zero.

*__MULTIPLICATIVE INVERSE.__

For any 2 × 2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying the condition $ad - bc \neq 0$, the matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

satisfies $AA^{-1} = A^{-1}A = I$. In this case, we say that A is invertible with multiplicative inverse A^{-1}.

Remarks. **(1)** The quantity $ad - bc$ is known as **the determinant of the matrix *A*.** The result above says that any 2 × 2 matrix is invertible as long as it has non-zero determinant.

(2) If two 2 × 2 matrices A and B both have non-zero determinants, then it can be shown that the matrix product AB also has non-zero determinant. We also have $(AB)^{-1} = B^{-1}A^{-1}$.

Example 9.Recall **Example1**. It is easy to check that

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

.

Example 10.The matrix $\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ has determinant 0 and so is not invertible.

Example 11:
Consider the matrices

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

Then

$$A^{-1} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \quad \text{and} \quad B^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}.$$

Note also that

$$AB = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 18 & 33 \\ 31 & 57 \end{pmatrix}.$$

We have

$$(AB)^{-1} = \frac{1}{3}\begin{pmatrix} 57 & -33 \\ -31 & 18 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = B^{-1}A^{-1}$$

.

V.2.:APPLICATION TO LINEAR EQUATIONS

We now return to the problem first discussed in previuos section. Consider the two linear equations

$$ax + by = s,$$
$$cx + dy = t,$$

,where *a,b,c,d,s,t* ∈R are given and *x* and *y* are the unknowns. We can represent these two linear equations in matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}.$$

If ad-bc≠0, then the 2 × 2 matrix on the left hand side is invertible. It follows that left hand side is invertible. It follows that there exist real numbers α,β,γ,δ ϵR such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix},$$

giving the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}.$$

Example 12: Suppose that

$$x + \quad y = 32$$
$$3x + 2y = 70.,$$

The two linear equations can be represented in matrix form

$$\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 32 \\ 70 \end{pmatrix}.$$

Note now that the matrix $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ has determinant −1 and multiplicative inverse : $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

It follows that,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 32 \\ 70 \end{pmatrix} = \begin{pmatrix} 6 \\ 26 \end{pmatrix}$$

$x = 6$ and $y = 26$.

ACTIVITIES FOR CHAPTER V

ACTIVITY V.1: Write each of the following systems of linear equations in matrix form:

a) $3x - 8y = 1$
$2x + 3y = 9$

b) $4x - 3y = 14$
$9x - 4y = 26$

c) $6x - 2y = 14$
$2x + 3y = 12$

d) $5x + 2y = 4$
$7x + 3y = 5$

ACTIVITY V.2: Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 3 \\ 5 & 1 \end{pmatrix}$$

a) Verify that $(A + B) + C = A + (B + C)$ and $A + B = B + A$.

b) Find $A + 3B$ and $7A - 2B + 3C$.

c) Verify that $(AB)C = A(BC)$.

d) Is it true that $AB = BA$? Comment on the result.

e) Find A^{-1} and B^{-1}.

f) Find $(AB)^{-1}$, and verify that $(AB)^{-1} = B^{-1}A^{-1}$.

ACTIVITY V.3: Solve each of the systems of linear equations in Question 1 by using inverse matrices

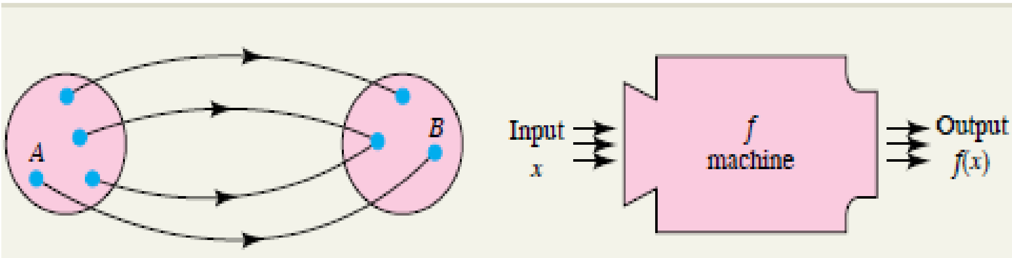
CHAPTER VI: FUNCTIONS AND GRAPHS

INTRODUCTION

The function concept is one of the most important ideas in mathematics. The study of mathematics beyond the elementary level requires a firm understanding of a basic list of elementary functions, their properties, and their graphs. Then we will learn how to apply them to different models in our everyday life.

VI.1: FUNCTIONS

A function is a correspondence between two sets of elements such that to each element in the first set, there corresponds one and only one element in the second set (or a **function** is a rule that assigns to each object in a set *A* exactly one object in a set *B*-***see figure 1)***. The set *A* is called the **domain** of the function, and the set of assigned objects in *B* is called the **range.**



(a) A function as a mapping (b) A function as a machine

Figure 1: Interpretations of the function *f*(*x*).

Tables 1 and **2** specify functions since to each domain value, there corresponds exactly one range value (for example, the cube of -2 is -8 and no other number). On the other hand, Table 3 does not specify a function since to at least one domain value, there corresponds more than one range value (for example, to the domain value 9, there corresponds -3 and 3, both square roots of 9).

Table 1

Domain *Number*	Range *Cube*
−2	−8
−1	−1
0	0
1	1
2	8

;

Table 2

Domain *Number*	Range *Square*
−2	4
−1	1
0	0
1	1
2	4

;

Table 3

Domain *Number*	Range *Square root*
0	0
1	1, −1
4	2, −2
9	3, −3

For most functions, the domain and range will be collections of real numbers and the function itself will be denoted by a letter such as *f*. The value that the function *f* assigns to the number *x* in the domain is then denoted by *f* (*x*) (read as"*f* of *x*"), which is often given by formulas, such as :

$$f(x) = x^2 + 4.$$

More complicated equations in two variables, such as $y = 9 - x^2$ or $x^2 = y^4$, are more difficult to graph. To **sketch the graph** of an equation, we plot enough points from its solution set in a rectangular coordinate system so that the total graph is apparent, and then we connect these points with a smooth curve. This process is called **point-by-point plotting**. Let us sketch the previous equations:

A) $f(x) = y = 9 - x^2$

Make up a table of solutions—that is, ordered pairs of real numbers that satisfy the given equation. For easy mental calculation, choose integer values for *x*.

x	−4	−3	−2	−1	0	1	2	3	4
y	−7	0	5	8	9	8	5	0	−7

After plotting these solutions, if there are any portions of the graph that are unclear, plot additional points until the shape of the graph is apparent. Then join all the plotted points with a smooth curve **(Fig. 2**). Arrowheads are used to indicate that the graph continues beyond the portion shown here with no significant changes in shape.

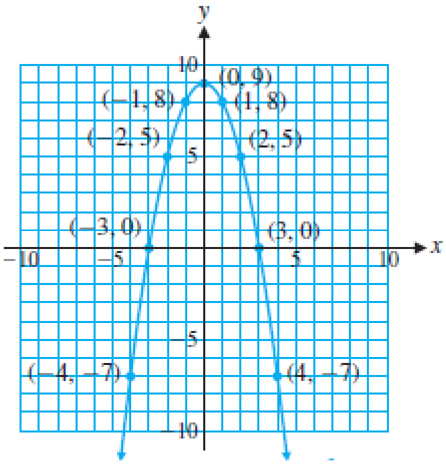


Figure 2: Graph of $y = 9 - x^2$

(B) $x^2 = y^4$

Again we make a table of solutions—here it may be easier to choose integer values for *y* and calculate values for *x*. Note, for example, that if $y = 2$, then $x = \{4$; that is, the ordered pairs (4, 2) and (-4, 2) are both in the solution set.

x	±9	±4	±1	0	±1	±4	±9
y	−3	−2	−1	0	1	2	3

We plot these points and join them with a smooth curve (**Fig. 3**).

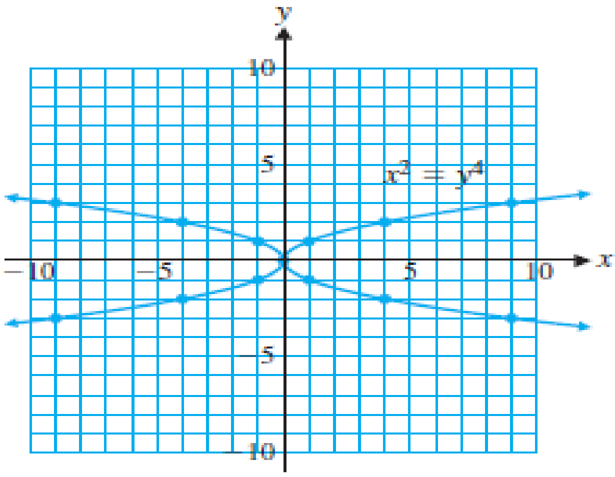


Figure 3: $x^2 = y^4$

The **input values** are **domain values**, and **the output values** are **range values**. The equation assigns each domain value *x* a range value *y*. The variable *x* is called an ***independent variable*** (since values can be "independently" assigned to *x* from the domain), and *y* is called a ***dependent variable*** (since the value of *y* "depends" on the value assigned to *x*). In general, any variable used as a placeholder for domain values is called an **independent variable**; any variable that is used as a placeholder for range values is called a **dependent variable**.

If in an equation in two variables, we get exactly one output (value for the dependent variable) for each input (value for the independent variable), then the equation, specifies **a function**. The graph of such a function is just the graph of the specifying equation. If **we get more than one output for a given input,** the equation does not specify a function.

Example 1: Determine which of the following equations specify functions with independent variable *x*.
(**A**) $4y - 3x = 8$, *x* a real number (**B**) $y^2 - x^2 = 9$, *x* a real number
Solution
(A) Solving for the dependent variable *y*, we have

$$\begin{aligned} 4y - 3x &= 8 \\ 4y &= 8 + 3x \qquad (1) \\ y &= 2 + \frac{3}{4}x \end{aligned}$$

Since each input value *x* corresponds to exactly one output value ($y = 2 + 3/4x$), we see that **equation (1) specifies a function**.
(**B**) Solving for the dependent variable *y*, we have

$$\begin{aligned} y^2 - x^2 &= 9 \\ y^2 &= 9 + x^2 \qquad (2) \\ y &= \pm\sqrt{9 + x^2} \end{aligned}$$

Since $9 + x^2$ is always a positive real number for any real number *x*, and since,each positive real number has two square roots,* then to each input value *x* there corresponds two output values :

$$\left(y = -\sqrt{9 + x^2} \text{ and } y = \sqrt{9 + x^2}\right)$$

For example, if $x = 4$, then equation (2) is satisfied for $y = 5$ and for $y = -5$. So equation **(2)** does not specify a function.

Since the **graph of an equation** is the graph of all the ordered pairs that satisfy the equation, it is very easy to determine whether an equation specifies a function by examining its graph. The graphs of the two equations we considered in **example 1** are shown in **figure 4.** In figure 4 **A**, we notice that any vertical line will intersect the graph of the equation $4y - 3x = 8$ in exactly one point. This shows that to each *x* value, there corresponds exactly one *y* value, confirming our conclusion that this equation specifies a function. On the other hand, Figure **4B** shows that there exist vertical lines that intersect the graph of $y^2 - x^2 = 9$ in two points. This indicates that there exist *x* values to which there correspond two different *y* values and verifies our conclusion that this equation does not specify a function:

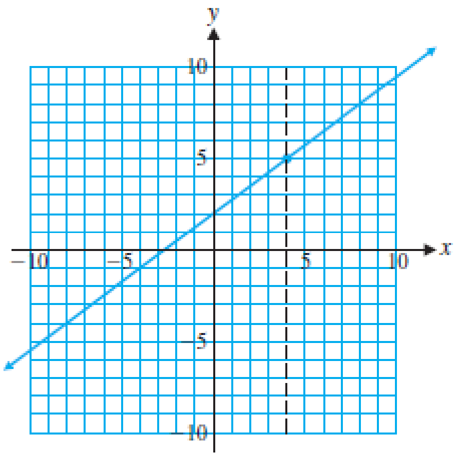


Figure 4 A

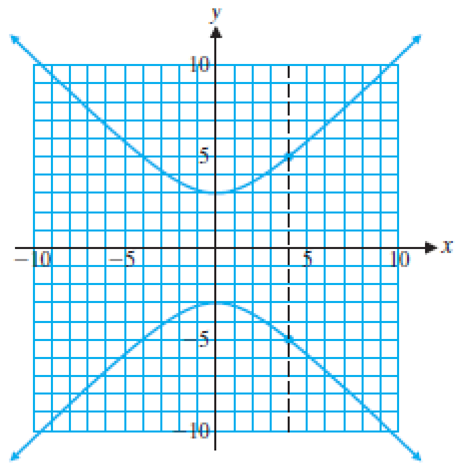


Figure 4 B

These observations are generalized in this **theorem- vertical-line test for a function**:

"An **equation specifies a function** if each vertical line in the coordinate system passes through, at most, one point on the graph of the equation.If any vertical line passes through two or more points on the graph of an equation, **then the equation does not specify a function**"

Functions are often defined **using more than one formula**, where each individual formula describes the function on a subset of the domain. A function defined in this way is sometimes called a **piecewise-defined function.** Here is an example of such a function:

Example 2:

Find $f\left(-\frac{1}{2}\right)$, $f(1)$, and $f(2)$ if

$$f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x < 1 \\ 3x^2 + 1 & \text{if } x \geq 1 \end{cases}$$

Since $x = -\frac{1}{2}$ satisfies $x < 1$, use the top part of the formula to find

$$f\left(-\frac{1}{2}\right) = \frac{1}{-1/2 - 1} = \frac{1}{-3/2} = -\frac{2}{3}$$

However, $x = 1$ and $x = 2$ satisfy $x \geq 1$, so $f(1)$ and $f(2)$ are both found by using the bottom part of the formula:

$$f(1) = 3(1)^2 + 1 = 4 \qquad \text{and} \qquad f(2) = 3(2)^2 + 1 = 13$$

Unless otherwise specified, if a formula (or several formulas-see above) is used to define a function *f*, then we assume the domain of *f* to be the set of all numbers for which *f*(*x*) is defined (as a real number). We refer to this as the natural domain of *f*.

Determining the natural domain of a function often amounts to excluding all numbers *x* that result in dividing by 0 or in taking the square root of a negative number. This procedure is illustrated in **example 3:**

Find the domain and range of each of these functions.

a. $f(x) = \frac{1}{x-3}$ **b.** $g(t) = \sqrt{t-2}$

Solution

a. Since division by any number other than 0 is possible, the domain of *f* is the set of all numbers *x* such that $x-3 \neq 0$; that is, $x \neq 3$. The range of *f* is the set of all numbers *y* except 0, since for any $y \neq 0$, there is an *x* such that in particular,

$$x = 3 + \frac{1}{y}.$$

b.Since negative numbers do not have real square roots, *g*(*t*) can be evaluated only when $t - 2 \geq 0$ so the domain of *g* is the set of all numbers *t* such that $t \geq 2$.The range of *g* is the set of all nonnegative numbers, for if $y \geq 0$ is any such number, there is a *t* such that,

$$y = \sqrt{t-2}; \text{ namely, } t = y^2 + 2.$$

VI.2: FUNCTIONS USED IN ECONOMICS

There are several functions associated with the marketing of a particular commodity:

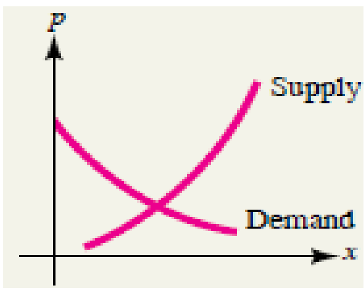
The demand function *D*(*x*) for the commodity is the price ***p*= *D*(*x*)** that must be charged for each unit of the commodity if *x* units are to be sold (demanded).

The supply function *S*(*x*) for the commodity is the unit price ***p*=*S*(*x*)** at which producers are willing to supply *x* units to the market. **The revenue *R*(*x*)** obtained from selling *x* units of the commodity is given by the product: ***R*(*x*)** = (number of items sold) (price per item)**=***xp*(*x*)

The cost function *C*(*x*) is the cost of producing *x* units of the commodity. **The profit function *P*(*x*)** is the profit obtained from selling *x* units of the commodity and is given by the difference:

$$P(x) = \text{revenue} - \text{cost}$$
$$= R(x) - C(x) = xp(x) - C(x)$$

Generally speaking, the higher the unit price, the fewer the number of units demanded, and vice versa. Conversely, an increase in unit price leads to an increase in the number of units supplied. Thus, demand functions are typically decreasing ("falling" from left to right), while supply functions are increasing ("rising"), as illustrated below:



Example:

Market research indicates that consumers will buy x thousand units of a particular kind of coffee maker when the unit price is

$$p(x)=-0.27x-51 \text{ dollars.}$$

The cost of producing the x thousand units is

$$C(x)=2.23x^2+3.5x+85 \text{ thousand dollars.}$$

a. What are **the revenue** and **profit** functions, $R(x)$ and $P(x)$, for this production process?

b. For what values of x is production of the coffee makers profitable?

Solution

a. The revenue is

$$R(x) = xp(x) = -0.27x^2 + 51x$$

thousand dollars, and the profit is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.27x^2 + 51x - (2.23x^2 + 3.5x + 85) \\ &= -2.5x^2 + 47.5x - 85 \end{aligned}$$

thousand dollars.

b. Production is profitable when $P(x) > 0$. We find that

$$\begin{aligned} P(x) &= -2.5x^2 + 47.5x - 85 \\ &= -2.5(x^2 - 19x + 34) \\ &= -2.5(x - 2)(x - 17) \end{aligned}$$

Since the coefficient -2.5 is negative, it follows that $P(x) > 0$ only if the terms $(x - 2)$ and $(x - 17)$ have different signs; that is, when $x - 2 > 0$ and $x - 17 < 0$. Thus, production is profitable for $2 < x < 17$.

CHAPTER VII: INTRODUCTION TO LIMITS

INTRODUCTION

How do algebra and calculus differ? The two words *static* and *dynamic* probably come as close as any to expressing the difference between the two disciplines. In algebra, we solve equations for a particular value of a variable—a static notion. In calculus, we are interested in how a change in **one variable** affects **another variable**—a dynamic notion.**Isaac Newton** (1642–1727) of England and **Gottfried Wilhelm von Leibniz** (1646–1716) of Germany developed calculus independently to solve problems concerning motion. Today calculus is used not just in the physical sciences, but also in business, economics, life sciences, and social sciences—any discipline that seeks to understand dynamic phenomena. We introduce the *derivative* and the *integral*, the two key concepts of **calculus**. Both key concepts depend on the notion of *limit*, explained here and will consider many applications of limits and derivatives.

VII.1: BRIEF REVIEW OF FUNCTIONS GRAPHS

The graph of the function $y = f(x) = x + 2$ is the graph of the set of all ordered pairs $(x, f(x))$. **The figure 1** shows the ordered pairs

$$(-1, f(-1)),\ (1, f(1)),\ \text{and}\ (2, f(2))$$

plotted on the graph of *f:*

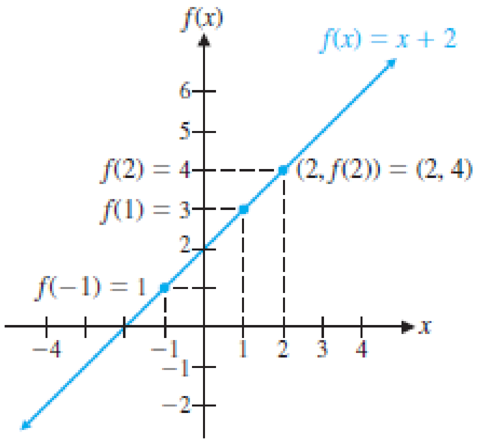
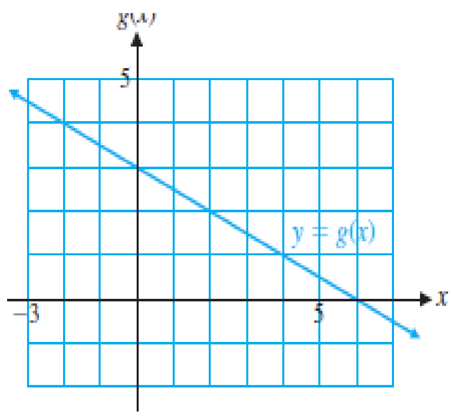


Figure 1: ***Graph of y = f(x) = x + 2***

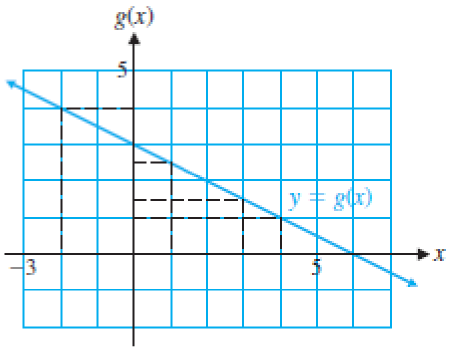
Now let us find of a function from Its Graph in completing the following table, using the given graph of the function *g*.

x	$g(x)$
−2	
1	
3	
4	

;



To determine ***g*(*x*)**, proceed vertically from the *x* value on the *x* axis to the graph of *g* and then horizontally to the corresponding ***y*** value *g*(*x*) on the *y* axis (as indicated by the dashed lines).



x	$g(x)$
−2	4.0
1	2.5
3	1.5
4	1.0

Matched Problem: Complete the following table, using the given graph of the function *h*.

VII.2-LIMITS: A GRAPHICAL APPROACH

We introduce the important concept of a *limit* through an example, which leads to an intuitive definition of the concept.

Let analyse a Limit of $f(x) = x + 2$ and **discuss** the behavior of the values of $f(x)$ when x is close to 2. We begin by drawing a graph of *f* that includes the domain value $x = 2$ (**fig. 2**):

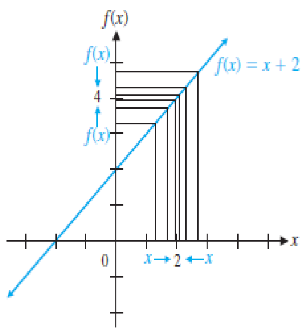


Figure 2: Domain value x=2 in f(x)= *x* + 2

In Figure 2, we are using a static drawing to describe a dynamic process. This requires careful interpretation. The thin vertical lines in Figure 2 represent values of *x* that are close to 2. The corresponding horizontal lines identify the value of $f(x)$ associated with each value of *x.* [the previous example dealt with the relationship between *x* and $f(x)$ on a graph.] The graph in **Figure 2** indicates that as the values of *x* get closer and closer to **2** on either side of 2, the corresponding values of $f(x)$ get closer and closer to 4. Symbolically, we write

$$\lim_{x\to 2} f(x) = 4$$

This equation is read as “The limit of $f(x)$ as *x* approaches 2 is 4.” Note that $f(x) = 4$. That is, the value of the function at **2** and the limit of the function as *x* approaches 2 are the same. This relationship can be expressed as

$$\lim_{x\to 2} f(x) = f(2) = 4$$

Graphically, this means that there is no hole or break in the graph of *f* at $x = 2$.

Matched Problem: Let *f*(*x*) = *x* + 1. Discuss the behavior of the values of *f*(*x*) when *x* is close to 1.

We now present an **informal definition of the important concept of a limit and** will write**:**

$$\lim_{x \to c} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to c$$

if **the functional value** *f*(*x*) is close to the single real number *L* whenever *x* is close, but not equal, to *c* (on either side of *c*).

Note: The existence of a limit at *c* has nothing to do with the value of the function at *c*. In fact, *c* may not even be in the domain of *f*. However, the function must be defined on both sides of *c*.

*To **make the informal definition of *limit* precise**, we must make the word *close* more precise. This is done as follows: We write

We write $\lim_{x \to c} f(x) = L$ if, for each $e > 0$, there exists a $d > 0$ such that $|f(x) - L| < e$ whenever $0 < |x - c| < d$.'

This definition is used to establish particular limits and to prove many useful properties of limits that will be helpful in finding particular limits.

The next example involves the **absolute value function**:

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \qquad \begin{aligned} f(-2) &= |-2| = -(-2) = 2 \\ f(3) &= |3| = 3 \end{aligned}$$

The graph of *f* is shown in figure **3 below:**

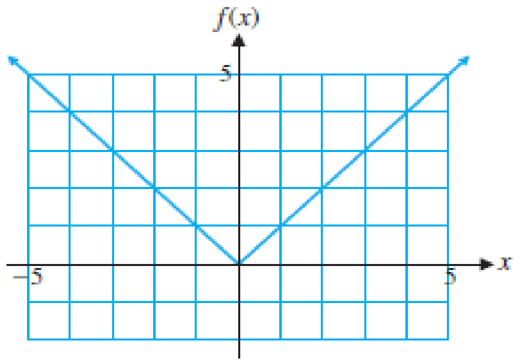


Figure 3: *f* (*x*) = | *x* |

Let analyze a limit of

$h(x) = |x|/x$ and explore the behavior of h(*x*) for *x* near, but not equal, to 0. We have to find $\lim_{x \to 0} h(x)$ if it exists.

The function *h* is defined for all real numbers except 0 $[h(0) = |0|/0 \text{ is undefined}]$

For example,

$$h(-2) = \frac{|-2|}{-2} = \frac{2}{-2} = -1$$

Note that if *x* is any negative number, then *h*(*x*) = -1if ***x*** < **0**, then the numerator **|** ***x*** **|** is positive but the denominator *x* is negative, so

$$h(x) = |x|/x = -1)$$

If *x* is any positive number, then *h*(*x*)= 1 (if *x* > 0, then the numerator**|** ***x*** **|** is equal to the denominator *x*,

$$h(x) = |x|/x = 1).$$

Figure 4 illustrates the behavior of *h*(*x*) for *x* near 0**:**

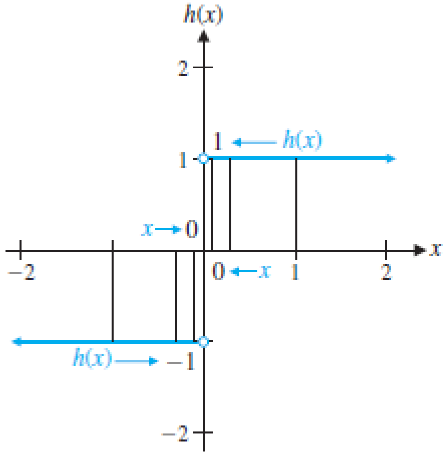


Figure 4: h(x)= **| *x* |/x**

Note that the absence of a solid dot on the vertical axis indicates that *h* is not defined when *x* = 0.When *x* is near 0 (on either side of 0), is *h*(*x*) near one specific number? The answer is "No," because *h*(*x*) is -1 for $x < 0$ and 1 for $x > 0$. Consequently, we say that

$$\lim_{x \to 0} \frac{|x|}{x} \text{ does not exist}$$

Neither *h*(*x*) nor the limit of *h*(*x*) exists at *x* = 0. However, the limit from the left and the limit from the right both exist at 0, but they are not equal.

Matched Problem:

Graph

$$h(x) = \frac{x - 2}{|x - 2|}$$

and find $\lim_{x \to 2} h(x)$ if it exists

In previous example, we see that the values of the function *h*(*x*) approach two different numbers, depending on the direction of approach, and it is natural to refer to these values as "the limit from the left" and "the limit from the right." These experiences suggest that the notion of **one-sided limits** will be very useful in discussing basic limit concepts.If no direction is specified in a limit statement, we will always assume that the limit is **two-sided** or **unrestricted**.

" For a (two-sided) limit to exist, the limit from the left and the limit from the right must exist and be equal. That is,

$$\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

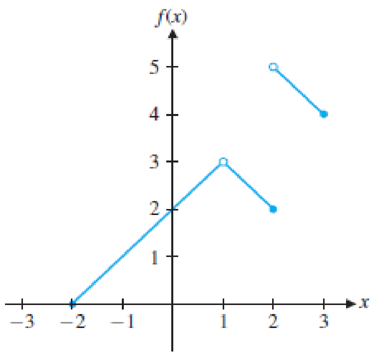
In citated example, since the left- and right-hand limits are *not* the same:

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|x|}{x} = 1$$

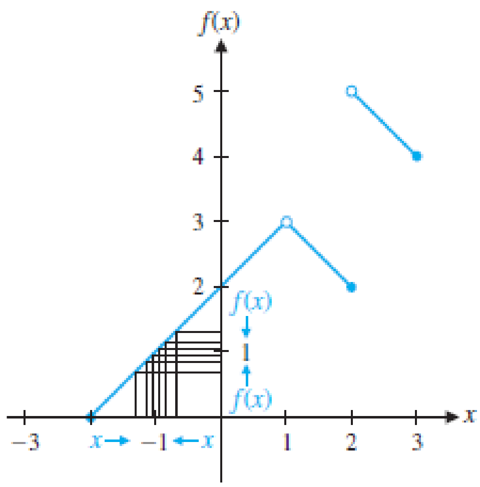
$$\lim_{x \to 0} \frac{|x|}{x} \text{ does not exist}$$

Example 4-Analyzing Limits Graphically:

Given the graph of the function f in **figure 5**, discuss the behavior of $f(x)$ for x near (A) -1, (B) 1, and (C) 2.

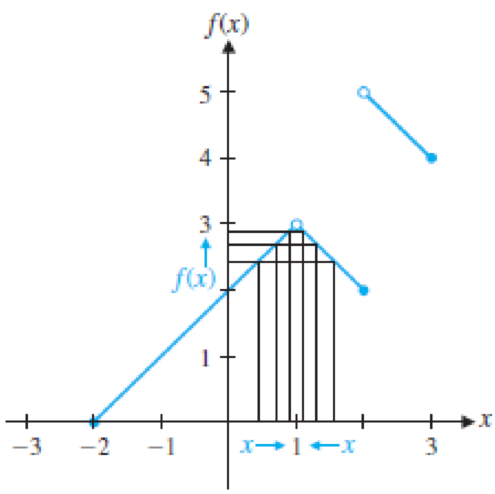


A)Since we have only a graph to work with, we use vertical and horizontal lines to relate the values of x and the corresponding values of $f(x)$. For any x near -1on either side of -1, we see that the corresponding value of $f(x)$, determined by a horizontal line, is close to 1.



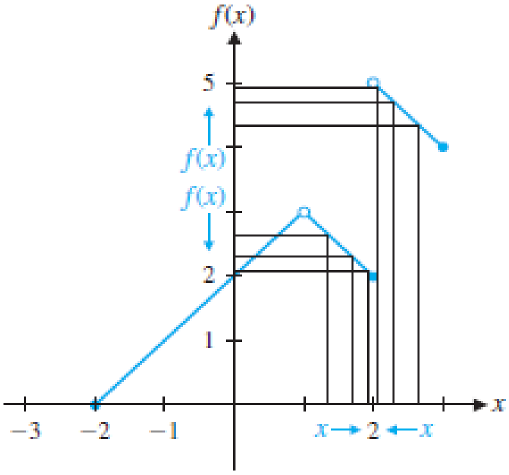
$$\lim_{x \to -1^-} f(x) = 1$$
$$\lim_{x \to -1^+} f(x) = 1$$
$$\lim_{x \to -1} f(x) = 1$$
$$f(-1) = 1$$

(B) Again, for any x near, but not equal to, 1, the vertical and horizontal lines indicate that the corresponding value of $f(x)$is close to 3. The open dot at (1, 3),together with the absence of a solid dot anywhere on the vertical line through $x = 1$, indicates that $f(1)$ is not defined.



$$\lim_{x \to 1^-} f(x) = 3$$
$$\lim_{x \to 1^+} f(x) = 3$$
$$\lim_{x \to 1} f(x) = 3$$
$$f(1) \text{ not defined}$$

(**C**) The abrupt break in the graph at *x* = 2 indicates that the behavior of the graph near *x* = 2 is more complicated than in the two preceding cases. If *x* is close to 2 on the left side of 2, the corresponding horizontal line intersects the *y* axis at a point close to 2. If *x* is close to 2 on the right side of 2, the corresponding horizontal line intersects the *y* axis at a point close to 5. This is a case where the one-sided limits are different.



$$\lim_{x\to 2^-} f(x) = 2$$
$$\lim_{x\to 2^+} f(x) = 5$$
$$\lim_{x\to 2} f(x) \text{ does not exist}$$
$$f(2) = 2$$

In **Example 4B,** note that $\lim_{x\to 1} f(x)$ exists even though *f* is not defined at *x* = 1 and the graph has a hole at *x* = 1. In general, the value of a function at *x* = *c* has no effect on the limit of the function as *x* approaches *c*.

VII.3-AN ALGEBRAIC APPROACH OF LIMITS

Graphs are very useful tools for investigating limits, especially if something unusual happens at the point in question. However, many of the limits encountered in calculus are routine and can be evaluated quickly with a little algebraic simplification, some intuition, and **basic properties of limits**. The following list of properties of limits forms the basis for this approach:

Let ***f*** and ***g*** be two functions, and assume that

$$\lim_{x\to c} f(x) = L \qquad \lim_{x\to c} g(x) = M$$

where *L* and *M* are real numbers (both limits exist). Then

1. $\lim_{x\to c} k = k$ for any constant k
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$
4. $\lim_{x\to c} [f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$
5. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) = kL$ for any constant k
6. $\lim_{x\to c} [f(x)\cdot g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)] = LM$
7. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ $L > 0$ for n even

Each property is also valid if $x \to c$ is replaced everywhere by $x \to c^-$ or replaced everywhere by $x \to c^+$.

EXAMPLE 5- USING LIMIT PROPERTIES

Find $\lim_{x\to 3}(x^2\text{-}4x)$

Solution

$$\lim_{x\to 3}(x^2 - 4x) = \lim_{x\to 3} x^2 - \lim_{x\to 3} 4x \quad \text{Property 4}$$

$$= \left(\lim_{x\to 3} x\right)\cdot\left(\lim_{x\to 3} x\right) - 4\lim_{x\to 3} x \quad \text{Properties 5 and 6}$$

$$= \left(\lim_{x\to 3} x\right)^2 - 4\lim_{x\to 3} x \quad \text{Definition of exponent}$$

$$= 3^2 - 4\cdot 3 = -3$$

So, omitting the steps in the dashed boxes,

$$\lim_{x\to 3}(x^2 - 4x) = 3^2 - 4\cdot 3 = -3$$

If $f(x) = x^2\text{-}4x$ and c is any real number, then, just as in example 5

$$\lim_{x\to c} f(x) = \lim_{x\to c}(x^2 - 4x) = c^2 - 4c = f(c)$$

So the limit can be found easily by evaluating the function f at c. This simple method for finding limits is very useful, because there are many functions that satisfy the property. This simple method for finding limits is very useful, because there are many functions that satisfy the property

$$\lim_{x\to c} f(x) = f(c)$$

Any polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

satisfies that property for any real number c. Also, any rational function

$$r(x) = \frac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomials, satisfies the previous property where $d(c) \neq 0$ and c is a real number and we will have:

1. $\lim_{x\to c} f(x) = f(c)$ for f any polynomial function.
2. $\lim_{x\to c} r(x) = r(c)$ for r any rational function with a nonzero denominator at $x = c$.

EXAMPLE 6- EVALUATING LIMITS

Find each limit.

(A) $\lim_{x\to 2}(x^3 - 5x - 1)$ (B) $\lim_{x\to -1}\sqrt{2x^2 + 3}$ (C) $\lim_{x\to 4}\frac{2x}{3x + 1}$

Solution:

(A) $\lim_{x\to 2}(x^3 - 5x - 1) = 2^3 - 5\cdot 2 - 1 = -3$

(B) $\lim_{x\to -1}\sqrt{2x^2 + 3} = \sqrt{\lim_{x\to -1}(2x^2 + 3)}$

(B) $\lim_{x \to -1} \sqrt{2x^2 + 3} = \sqrt{\lim_{x \to -1} (2x^2 + 3)}$

$$= \sqrt{2(-1)^2 + 3}$$

$$= \sqrt{5}$$

(C) $\lim_{x \to 4} \frac{2x}{3x + 1} = \frac{2 \cdot 4}{3 \cdot 4 + 1}$

$$= \frac{8}{13}$$

Matched Problem: Find each limit.

(A) $\lim_{x \to -1} (x^4 - 2x + 3)$ (B) $\lim_{x \to 2} \sqrt{3x^2 - 6}$ (C) $\lim_{x \to -2} \frac{x^2}{x^2 + 1}$

Example 7: Evaluating Limits

Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

Find:

(A) $\lim_{x \to 2^-} f(x)$ (B) $\lim_{x \to 2^+} f(x)$ (C) $\lim_{x \to 2} f(x)$ (D) $f(2)$

Solution:

(A) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 1)$ If $x < 2$, $f(x) = x^2 + 1$

$= 2^2 + 1 = 5$

(B) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 1)$ If $x > 2$, $f(x) = x - 1$.

$= 2 - 1 = 1$

(C) Since the one-sided limits are not equal, $\lim_{x \to 2} f(x)$ does not exist.

(D) Because the definition of *f* does not assign a value to *f* for $x = 2$, only for $x < 2$ and $x > 2$, $f(2)$does not exist.

Matched Problem:

Let

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 5 \\ -x + 12 & \text{if } x > 5 \end{cases}$$

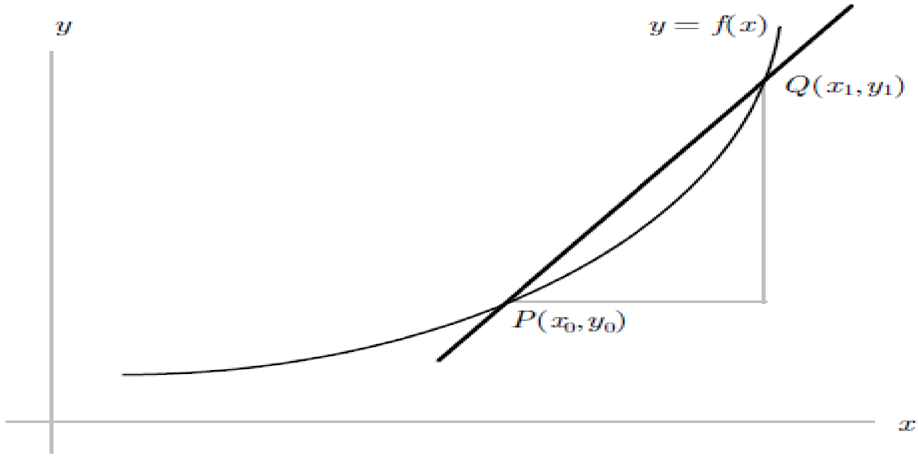
Find:

(A) $\lim_{x \to 5^-} f(x)$ (B) $\lim_{x \to 5^+} f(x)$ (C) $\lim_{x \to 5} f(x)$ (D) $f(5)$

CHAPTER VIII: INTRODUCTION TO DIFFERENTIATION

VIII.1: TANGENT TO A CURVE

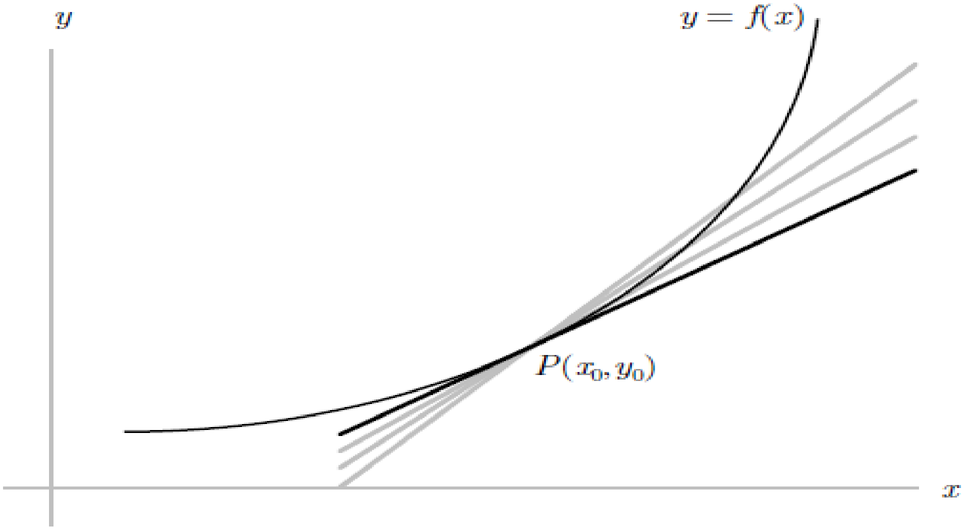
Consider the graph of a function $y = f(x)$. Suppose that $P\ (x_0,\ y_0)$ is a point on the curve $y = f(x)$. Consider now another point $Q\ (x_1,\ y_1)$ on the curve close to the point $P\ (x_0,\ y_0)$. We draw the line joining the points $P(x_0,\ y_0)$ and $Q(x_1,\ y_1)$, and obtain the picture below.



Clearly the slope of this line is equal to:

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Now let us keep the point $P(x_0,y_0)$ fixed, and move the point $Q(x_1,y_1)$ along the curve towards the point P. Eventually the line PQ becomes the tangent to the curve $y = f(x)$ at the point $P(x_0,y_0)$, as shown in the picture below.

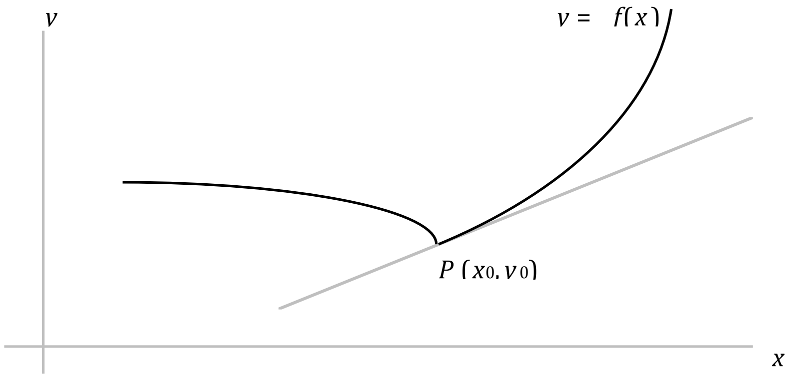


We are interested in the slope of this tangent line. Its value is called the derivative of the function $y = f(x)$ at the point $x = x_0$, and denoted by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} \qquad \text{or} \qquad f'(x_0).$$

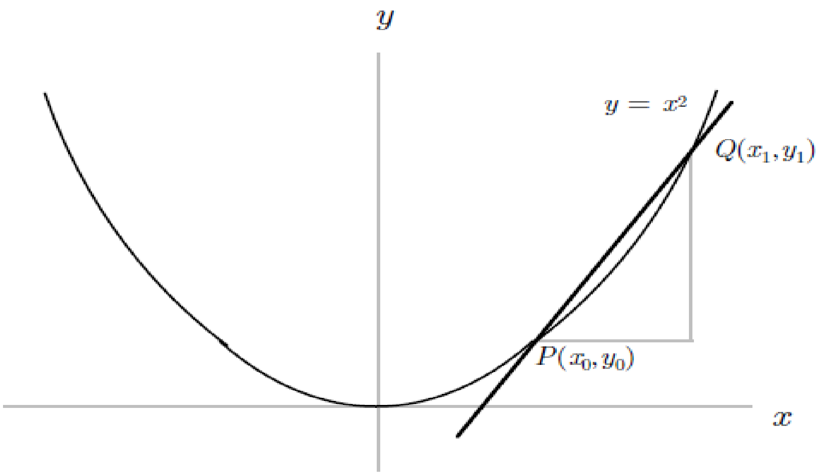
In this case, we say that the function $y = f(x)$ is differentiable at the point $x = x_0$.

Remark. Sometimes, when we move the point $Q\ (x_1, y_1)$ along the curve $y = f(x)$ towards the point $P(x_0,y_0)$, the line PQ does not become the tangent to the curve $y = f(x)$ at the point $P(x_0,y_0)$. In this case, we say that the function $y = f(x)$ is not differentiable at the point $x = x_0$. An example of such a situation is given in the picture below.



Note that the curve $y = f(x)$ makes an abrupt turn at the point $P(x_0,y_0)$.

Example 1: Consider the graph of the function $y = f(x) = x^2$.



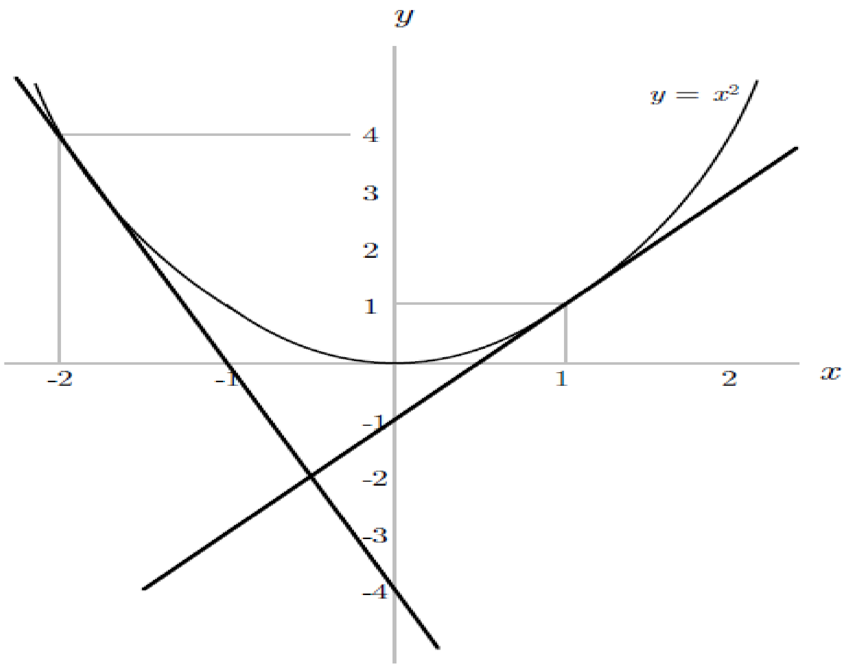
Here the slope of the line joining the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = x_1 + x_0$$

It follows that if we move the point $Q(x_1,y_1)$ along the curve towards the point $P(x_0,y_0)$, then the slope of this line will eventually be equal to $x_0 + x_0 = 2x_0$. Hence for the function $y = f(x) = x^2$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 2x_0.$$

In particular, the tangent to the curve at the point (1 1) has slope 2 and so has equation $y = 2x - 1$, whereas the tangent to the curve at the point (−2,4) has slope −4 and so has equation $y = -4x - 4$.



y

Example.2: Consider the graph of the function *y* = *f*(*x*) = x^3. Here the slope of the line joining the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is equal to:

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^3 - x_0^3}{x_1 - x_0} = x_1^2 + x_1 x_0 + x_0^2.$$

It follows that if we move the point $Q(x_1,y_1)$ along the curve towards the point $P(x_0,y_0)$, then the slope of this line will eventually be equal to:

$$x_0^2 + x_0 x_0 + x_0^2 = 3x_0^2.$$

. Hence for the function $y = f(x) = x^3$, we have:

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0) = 3x_0^2.$$

In particular, the tangent to the curve at the point (0,0) has slope 0 and so has equation $y = 0$, whereas the tangent to the curve at the point (2,8) has slope 12 and so has equation $y = 12x - 16$.

Example 3: Consider the graph of the function *y* = *f*(*x*) = *x*. Here the slope of the line joining the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = 1.$$

It follows that if we move the point $Q(x_1,y_1)$ along the curve towards the point $P(x_0,y_0)$, then the slope of this line will remain equal to 1. Hence for the function $y = f(x) = x$, we have

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0) = 1.$$

Example 4:Consider the graph of the function $y = f(x) = x^{1/2}$, defined for all real numbers $x \geq 0$. Suppose that $x_0 > 0$ and $x_1 > 0$. Then the slope of the line joining the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^{1/2} - x_0^{1/2}}{x_1 - x_0} = \frac{1}{x_1^{1/2} + x_0^{1/2}}.$$

It follows that if we move the point $Q(x_1,y_1)$ along the curve towards the point $P(x_0,y_0)$, then the slope of this line will eventually be equal to

$$\frac{1}{x_0^{1/2} + x_0^{1/2}} = \frac{1}{2x_0^{1/2}} = \frac{1}{2}x_0^{-1/2}.$$

Hence for the function $y = f(x) = x^{1/2}$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = \frac{1}{2}x_0^{-1/2}.$$

The above four examples are special cases of the following result.

VIII.2: DERIVATIVES OF POWERS.

Suppose that n is a fixed non-zero real number. Then for the function $y = f(x) = x^n$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = nx^{n-1}$$

for every real number x for which x^{n-1} is defined. Here and henceforth, we shall slightly abuse our notation and refer to f '(x) as the derivative of the function $y = f(x)$, and write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

Example :5 For the function $y = f(x) = x^{1/4}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \frac{1}{4}x^{-3/4}$$

for every positive real number x. The rule concerning derivatives of powers does not apply in the case $n = 0$.

Suppose that *f*(*x*) = *c*, where *c* is a fixed real number. **Then *f'*(*x*) = *0*** for every real number x.

VIII.3:ARITHMETIC OF DERIVATIVES

Very often, we need to find the derivatives of complicated functions which are constant multiples, sums, products and/or quotients of much simpler functions. To achieve this, we can make use of our knowledge concerning the derivatives of these simpler functions. We have four extremely useful results.

VIII.3.1: CONSTANT MULTIPLE RULE.

Suppose that $m(x) = cf(x)$, where c is a fixed real number. Then **m'(x)=c *f"*(*x*)** for every real number x for which ***f"*(*x*)** exists.

VIII.3.2: SUM RULE.

Suppose that $s(x) = f(x) + g(x)$ and $d(x) = f(x) - g(x)$. Then

$$s'(x) = f'(x) + g'(x) \quad \text{and} \quad d'(x) = f'(x) - g'(x)$$

for every real number x for which f'(x) and g'(x) exist.

Example2: Consider the function $h(x) = 5x^2 + 3x^5$. We can write: $h(x) = f(x) + g(x)$,

where $f(x) = 5x^2$ and $g(x) = 3x^5$. It follows from the sum rule that $h'(x) = f'(x) + g'(x)$.

Next, the function $f(x) = 5x^2$ is a constant (5) multiple of the function x^2, and so it follows from the **constant multiple rule and the rule on the derivatives of powers** that

$$f'(x) = 5(x^2)' = 5(2x) = 10x.$$

Similarly, the function $g(x) = 3x^5$ is a constant (3) multiple of the function x^5, and so it follows from the **constant multiple rule and the rule on the derivatives of powers** that

$$g'(x) = 3(x^5)' = 3(5x^4) = 15x^4.$$

Hence $h'(x) = 10x + 15x^4$.

$$h'(x) = 10x + 15x^4.$$

Example 3: Consider the function $h(x) = (3x)^4 - (2x)^6$. We can write: $h(x) = f(x) - g(x)$,

where $f(x) = 81x^4$ and $g(x) = 64x^6$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain

$f''(x)= 324x^3$ **and** $g(x)=384x^5$. Hence $h'(x) = 324x^3 - 384x^5$

The sum rule can be extended to the sum or difference of more than two functions in the natural way. We illustrate the technique in the following the examples.

Example 4: Consider the function $h(x) = (x^2 + 2x)^2$. Then $h(x) = x^4 + 4x^3 + 4x^2$, and so we can write $h(x) = f(x) + g(x) + k(x)$, where $f(x) = x^4$, $g(x) = 4x^3$ and $k(x) = 4x^2$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x) + k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain

$$h'(x) = 4x^3 + 12x^2 + 8x.$$

EXAMPLE 5: Consider the function

$$h(x) = \frac{3}{x} + 2x.$$

We can write: h(x) = f(x) + g(x), where f(x) = $3x^{-1}$ and g(x) = $2x$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain

$$h'(x) = 2 - 3x^{-2}.$$

EXAMPLE 6: Consider the function

$$h(x) = 6x^2\sqrt{x} - \frac{4}{\sqrt{x}} + 3x^{1/3}.$$

We can write $h(x) = f(x) - g(x) + k(x)$,where $f(x) = 6x^{5/2}$, $g(x) = 4x^{-1/2}$
and $k(x) = 3x^{1/3}$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x) + k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 15x^{3/2}$, $g'(x) = -2x^{-3/2}$ and $k'(x) = x^{-2/3}$. Hence:

$$h'(x) = 15x^{3/2} + 2x^{-3/2} + x^{-2/3}.$$

Example 7: Consider the function :

$$h(x) = \sqrt{3x} + \sqrt[3]{2x}.$$

We can write

$h(x) = f(x) + g(x)$, *Where*

$f(x) = \sqrt{3}x^{1/2}$ and $g(x) = \sqrt[3]{2}x^{1/3}$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain

$$f'(x) = \frac{\sqrt{3}}{2}x^{-1/2} \quad \text{and} \quad g'(x) = \frac{\sqrt[3]{2}}{3}x^{-2/3}.$$

Hence:

$$h'(x) = \frac{\sqrt{3}}{2}x^{-1/2} + \frac{\sqrt[3]{2}}{3}x^{-2/3} = \sqrt{\frac{3}{4x}} + \sqrt[3]{\frac{2}{27x^2}}.$$

VIII.3.3: PRODUCT RULE.

Suppose that $p(x) = f(x)g(x)$. Then $p'(x) = f'(x)g(x) + f(x)g'(x)$ for every real number x for which $f'(x)$ and $g'(x)$ exist.

Example 9: Consider the function $h(x) = (x^3 - x^5)(x^2 + x^4)$. We can write :$h(x) = f(x)g(x)$,

where $f(x) = x^3 - x^5$ and $g(x) = x^2 + x^4$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule and the rule on the derivatives of powers, we obtain $f'(x) = 3x^2 - 5x^4$ and $g'(x) = 2x + 4x^3$. Hence:

$$h'(x) = (3x^2 - 5x^4)(x^2 + x^4) + (x^3 - x^5)(2x + 4x^3) = 5x^4 - 9x^8.$$

Alternatively, we observe that $h(x) = (x^3-x^5)(x^2+x^4) = x^5-x^9$.Applying the sum rule and the rule on the derivatives of powers, we obtain

$$h'(x) = 5x^4 - 9x^8$$

as before.

The product rule can be extended to the product of more than two functions. The extension is at first sight somewhat less obvious than in the case of the sum rule. However, with a bit of care, it is in fact rather straightforward.

Example 10: Consider the function $h(x) = (x^2 + 4x)(2x + 1)(6 - 2x^2)$. We can write

$$h(x) = f(x)r(x),$$

where $f(x) = x^2 + 4x$ and $r(x) = (2x + 1)(6 - 2x^2)$. It follows from the product rule that

$$h'(x) = f'(x)r(x) + f(x)r'(x).$$

We can now write

$$r(x) = g(x)k(x),$$

where $g(x) = 2x + 1$ and $k(x) = 6 - 2x^2$. It follows from the product rule that

$$r'(x) = g'(x)k(x) + g(x)k'(x).$$

Hence $h(x) = f(x)g(x)k(x)$, and

$$h'(x) = f'(x)g(x)k(x) + f(x)g'(x)k(x) + f(x)g(x)k'(x).$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain:

$f'(x) = 2x + 4,\ g'(x) = 2$ and $k'(x) = -4x$. Hence:

$$h'(x) = (2x + 4)(2x + 1)(6 - 2x^2) + 2(x^2 + 4x)(6 - 2x^2) - 4x(x^2 + 4x)(2x + 1).$$

Remark. The interested reader is challenged to show that if $p(x) = f(x)g(x)k(x)t(x)$, then

$$p'(x) = f'(x)g(x)k(x)t(x) + f(x)g'(x)k(x)t(x) + f(x)g(x)k'(x)t(x) + f(x)g(x)k(x)t'(x).$$

VIII.3.4: QUOTIENT RULE.

Suppose that $q(x) = f(x)/g(x)$. Then:

$$q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

for every real number x for which ***f'*(*x*) *and g'*(*x*)** exist, and for which $g(x) \neq 0$

Example 11: Consider the function

$$h(x) = \frac{x^2 - 1}{x^3 + 2x}.$$

We can write:

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = x^2 - 1$ and $g(x) = x^3 + 2x$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain:

$f'(x) = 2x$ and $g'(x) = 3x^2 + 2$.

Hence:

$$h'(x) = \frac{2x(x^3 + 2x) - (x^2 - 1)(3x^2 + 2)}{(x^3 + 2x)^2}$$

Example 12: Consider the function:

$$h(x) = \frac{4x^2 + 1}{3x}.$$

where $f(x) = 4x^2 + 1$ and $g(x) = 3x$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 8x$ and $g'(x) = 3$. Hence:

$$h'(x) = \frac{24x^2 - 3(4x^2 + 1)}{9x^2}.$$

VIII.4: DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

Consider the curve $y = f(x) = \sin x$. Suppose that $P(x,f(x))$ is a point on this curve. Consider another point $Q(x+h,f(x+h))$, where $h \neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points P and Q is equal to:

$$\frac{f(x+h) - f(x)}{(x+h) - x} = \frac{\sin(x+h) - \sin x}{h}.$$

Consider the curve $y = g(x) = \cos x$. Suppose that $R(x,g(x))$ is a point on this curve. Consider another point $S(x+h,g(x+h))$, where $h \neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points R and S is equal to

$$\frac{g(x+h) - g(x)}{(x+h) - x} = \frac{\cos(x+h) - \cos x}{h}.$$

We now move the point Q along the curve $y = f(x) = \sin x$ towards the point P, and move the point S along the curve $y = g(x) = \cos x$ towards the point R.

$$\frac{\sin(x+h)-\sin x}{h} = \frac{\sin x\cos h+\cos x\sin h-\sin x}{h} = (\cos x)\times\frac{\sin h}{h}+(\sin x)\times\frac{\cos h-1}{h}$$

and

$$\frac{\cos(x+h)-\cos x}{h} = \frac{\cos x\cos h-\sin x\sin h-\cos x}{h} = -(\sin x)\times\frac{\sin h}{h}+(\cos x)\times\frac{\cos h-1}{h}$$

When h is very close to 0, then

$$\frac{\sin h}{h}\approx 1 \quad \text{and} \quad \frac{\cos h-1}{h}\approx 0,$$

so that

$$\frac{\sin(x+h)-\sin x}{h}\approx \cos x \quad \text{and} \quad \frac{\cos(x+h)-\cos x}{h}\approx -\sin x.$$

We have established the first two parts of the result below.

VIII.5: DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS.

***Suppose that *f*(*x*) = sinx and *g*(*x*) = cos*x* *and* *t*(*x*) = tanx.**

T(*x*) = **tan*x*=**$\frac{\boldsymbol{sinx}}{\boldsymbol{cosx}}$ = *f*(*x*)/*g*(*x*). It follows from the quotient rule that

$$t'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

→(tang x)'=sec²x

***Suppose that *t*(*x*) = cotx**. Then *t*(*x*) = *g*(*x*)/*f*(*x*)= $\frac{\boldsymbol{\cos x}}{\boldsymbol{sinx}}$. It follows from the quotient rule that:

$$t'(x) = \frac{f(x)g'(x)-g(x)f'(x)}{f^2(x)} = \frac{-\sin^2 x-\cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

→(cotx)'=-csc² *x*.

Suppose that *t*(*x*) = sec *x. Then *t*(*x*) = *c*(*x*)/*g*(*x*), where *c*(*x*) = 1.It follows from the quotient rule that

$$t'(x) = \frac{g(x)c'(x)-c(x)g'(x)}{g^2(x)} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\times\frac{1}{\cos x} = \tan x\sec x.$$

→(sec x)'= tan *x* sec *x*

***Suppose that *t*(*x*) = csc*x*. Then *t*(*x*) = *c*(*x*)/*f*(*x*).**It follows from the quotient rule that

$$t'(x) = \frac{f(x)c'(x)-c(x)f'(x)}{f^2(x)} = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\times\frac{1}{\sin x} = -\cot x\csc x.$$

→(csc x)'= -cot *x* csc *x*

Example 13: Consider the function *h*(*x*) = (*x*³ − 2) (sin *x* + cos *x*). We can write
h(*x*) = *f*(*x*)*g*(*x*), where *f*(*x*) = $x^3 - 2$ and *g*(*x*) = sin *x* + cos *x*. It follows that
h'(*x*) = *f'*(*x*)*g*(*x*) + *f*(*x*)*g'*(*x*).

Observe next that $f'(x) = 3x^2$ and $g'(x) = \cos x - \sin x$.

Hence $h'(x) = 3x^2(\sin x + \cos x) + (x^3 - 2)(\cos x - \sin x)$.

Example 14:Consider the function $\boldsymbol{h(x) = \frac{Sin\ x}{x}}$

We can write **$h(x) = f(x) / g(x)$**, where $f(x) = \sin x$ and $g(x) = x$. It follows that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = \cos x$ and $g'(x) = 1$. Hence

$$h'(x) = \frac{x \cos x - \sin x}{x^2}.$$

Example 15: **Consider the function $h(x) = \sin^2 x$.** We can write: $h(x) = f(x)g(x)$, where $f(x) = g(x) = \sin x$. It follows that: $h'(x) = f'(x)g(x) + f(x)g'(x)$. Observe next that $f'(x) = g'(x) = \cos x$.

Hence $h'(x) = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x$.

Example 16: Consider the function $y = \sin 2x$. We can write $h(x) = 2f(x)g(x)$, where $f(x) = \sin x$ and $g(x) = \cos x$. It follows that $h'(x) = 2(f'(x)g(x) + f(x)g'(x))$. Observe next that $f'(x) = \cos x$ and $g'(x) = -\sin x$. Hence $h'(x) = 2(\cos^2 x - \sin^2 x) = 2 \cos 2x$.

ACTIVITES FOR CHAPTER VIII

ACTIVITY VIII.1: For each of the following functions *f*(*x*), write down the derivative *f'*(*x*) as a function of *x*, and find the slope of the tangent at the point *P*(1,*f*(1))

a) $f(x) = x^4$ b) $f(x) = 5x^2$

c) $f(x) = \frac{1}{6}x^{-3}$ d) $f(x) = \pi x^{1.5}$

ACTIVITY VIII.2: Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants and sums:

a) $h(x) = 6x^3$ b) $h(x) = 5x^{-7}$
c) $h(x) = 12x - 3x^2$ d) $h(x) = x^3 + 4x$
e) $h(x) = 6x^2 - 40x$ f) $h(x) = x^7 + 6x^5 - 8x^2 + 3x$

ACTIVITY VIII.3: Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums and products as appropriate:

a) $h(x) = (x^2 + 3)(2x - 5)$ b) $h(x) = (x^2 - x + 2)(x^2 - 2)$
c) $h(x) = (x^2 + 5)(x^3 - 4x^2)$ d) $h(x) = (x^4 - 3x^3 + 2x)(3x^2 + 4x)$
e) $h(x) = (x^9 + 2x^3)x^{-4}$ f) $h(x) = (x^4 - 2x^3 + 7x + 8)^2$

ACTIVITY VIII.4: Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums, products and quotients as appropriate:

a) $h(x) = \frac{1}{x^4 + x^3 + 1}$ b) $h(x) = 1 + \frac{3}{x} - \frac{2}{x^2}$ c) $h(x) = \frac{x - 2}{x + 1}$
d) $h(x) = \frac{1 + x^2}{1 - x^2}$ e) $h(x) = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$ f) $h(x) = \frac{x}{x + x^{-1}}$

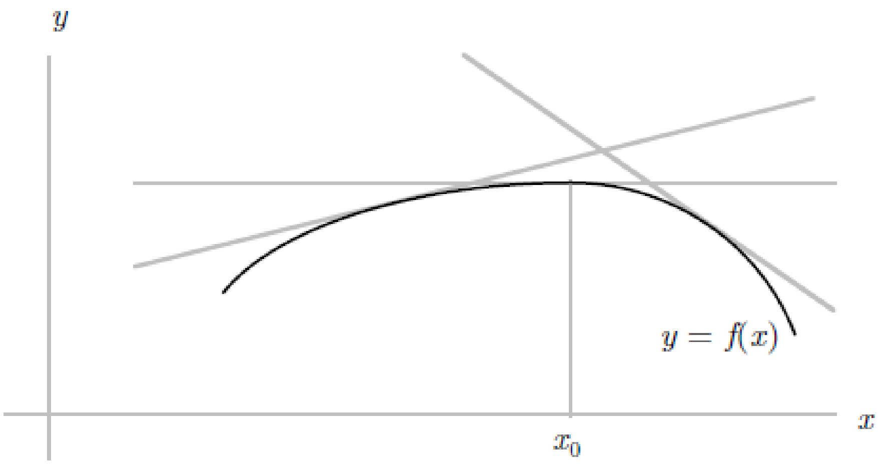
CHAPTER IX: APPLICATIONS OF DIFFERENTIATION

IX.1: SECOND DERIVATIVES

Recall that for a function $y = f(x)$, the derivative $f'(x)$ represents the slope of the tangent. It is easy to see from a picture that if the derivative $f'(x) > 0$, then the function $f(x)$ is increasing; in other words,$f(x)$ increases in value as x increases. On the other hand, if the derivative $f'(x) < 0$, then the function $f(x)$ is decreasing; in other words, $f(x)$ decreases in value as x increases. We are interested in the case when the derivative $f'(x) = 0$. **Values $x = x_0$ such that $f'(x_0) = 0$ are called stationary points**.

Let us introduce the second derivative $f''(x)$ of the function $f(x)$. This is defined to be the derivative of the derivative $f'(x)$. With the same reasoning as before but applied to the function $f'(x)$ instead of the function $f(x)$, we conclude that if the second derivative $f''(x) > 0$, then the derivative $f'(x)$ is increasing. Similarly, if the second derivative $f''(x) < 0$, then the derivative $f'(x)$ is decreasing.

Suppose that $f(x_0) = 0$ and $f'(x_0) < 0$. The condition $f''(x_0) < 0$ tells us that the derivative $f'(x)$ is decreasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f(x) > 0$ when x is a little smaller than x_0, and that $f'(x) < 0$ when x is a little greater than x_0, as indicated in the picture below.



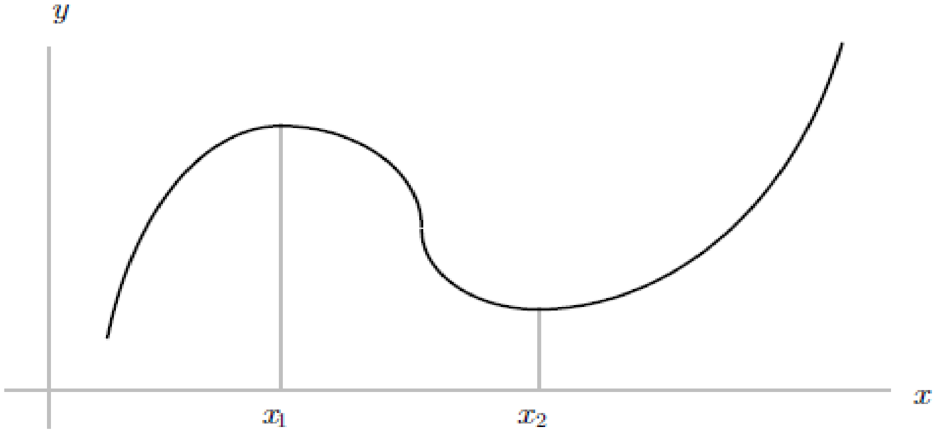
In this case, we say that the function has a **local maximum** at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \leq f(x_0)$ for all such real values x.

LOCAL MAXIMUM. *Suppose that* $f'(x_0) = 0$ *and* $f'(x_0) < 0$*. Then the function* $f(x)$ *has a local maximum at the point* $x = x_0$*.*

Suppose next that $f'(x_0) = 0$ and $f''(x_0) > 0$. The condition $f''(x_0) > 0$ tells us that the derivative $f'(x)$ is increasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f''(x) < 0$ when x is a little smaller than x_0, and that $f'(x) > 0$ when x is a little greater than x_0, as indicated in the picture below. In this case, we say that the function has a local minimum at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \geq f(x_0)$ for all such real values x.

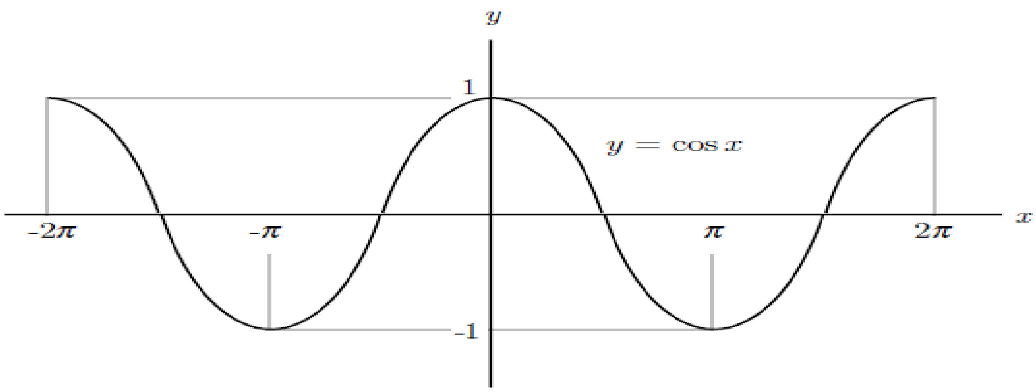
LOCAL MINIMUM. *Suppose that* $f'(x_0) = 0$ *and* $f''(x_0) > 0$*. Then the function* $f(x)$ *has a local minimum at the point* $x = x_0$*.*

Remark. These stationary points are called local maxima or local minima because such points may not maximize or minimize the functions in question. Consider the picture below, with a local maximum at $x = x_1$ and a local minimum at $x = x_2$.



We also say that a point $x = x_0$ is a point of inflection if $f''(x_0) = 0$, irrespective of whether $f'(x_0) = 0$ or not. A simple way of visualizing the graph of a function at a point of inflection is to imagine that one is steering a car along the curve. A point of inflection then corresponds to the place on the curve where the steering wheel of the car is momentarily straight while being turned from a little left to a little right, or while being turned from a little right to a little left.

Example 1: Consider the function $f(x) = \cos x$. Since $f'(x) = -\sin x = 0$ whenever $x = k\pi$, where $k \in Z$, it follows that the function $f(x) = \cos x$ has a stationary point at $x = k\pi$ for every $k \in Z$. Next, note that $f''(x) = -\cos x$. If k is even, then $f''(k\pi) = -1$, so that $f(x)$ has a local maximum at $x = k\pi$. If k is odd, then $f''(k\pi) = 1$, so that $f(x)$ has a local minimum at $x = k$:.see the graph below:



Example 2: **Consider the function** $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$. Since $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x - 1)(x + 2)$, it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -2$. On the other hand, we have $f''(x) = 36x^2 + 24x - 24$. Since $f''(0) = -24$, $f''(1) = 36$ and $f''(-2) = 72$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -2$.

Example 3: Consider the function $f(x) = x^3 - 3x^2 + 2$. Since $f'(x) = 3x^2 - 6x = 3x(x - 2)$, it follows that **the function** $f(x)$ **has stationary points at** $x = 0$ **and** $x = 2$. On the other hand, we have

$f''(x) = 6x - 6$. Since $f''(0) = -6$ and $f''(2) = 6$, it follows that $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$. Observe also that there is a point of inflection at $x = 1$.

Example 4: Consider the function $f(x) = x^4 - 2x^2 + 7$. Since $f''(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1)$,it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -1$. On the other hand, we have $f''(x) = 12x^2 - 4$. Since $f''(0) = -4$, $f''(1) = 8$ and $f''(-1) = 8$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -1$. Note also that $f(x) = 0$ if $x = \pm\sqrt{1}/3$,so there are points of inflection at $x = \pm\sqrt{1}/3$

Example 5: Consider the function $f(x) = \sin x - \cos 2x$, restricted to the interval $0 \le x \le 2\pi$.
It is easy to see that $f''(x) = \cos x + 2\cos x \sin x = (1 + 2\sin x)\cos x$. We therefore have stationary points when $\cos x = 0$ or $\sin x = -1/2$. There are four stationary points in the interval $0 \le x \le 2\pi$, namely

$$x = \frac{\pi}{2}, \qquad x = \frac{3\pi}{2}, \qquad x = \frac{7\pi}{6}, \qquad x = \frac{11\pi}{6}.$$

Next, note that we can write $f'(x) = \cos x + \sin 2x$, so that $f\_\_(x) = 2\cos 2x - \sin x$. It is easy to check that

$$f''\left(\frac{\pi}{2}\right) = -3, \qquad f''\left(\frac{3\pi}{2}\right) = -1, \qquad f''\left(\frac{7\pi}{6}\right) = \frac{3}{2}, \qquad f''\left(\frac{11\pi}{6}\right) = \frac{3}{2}.$$

Hence $f(x)$ has local maxima at $x = \pi/2$ and $x = 3\pi/2$, and local minima at $x = 7\pi/6$ and $x = 11\pi/6$.

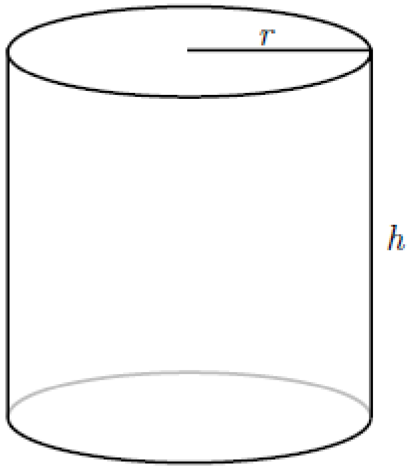
IX.2: APPLICATIONS TO PROBLEM SOLVING.

In this section, we discuss how we can apply ideas in differentiation to solve various problems. We shall illustrate the techniques by discussing a few examples. Central to all of these is the crucial step where we set up the problems mathematically and in a suitable way.

Example 1: We wish to find positive real numbers x and y such that $x+y = 6$ and the quantity xy^2 is as large as possible. In view of the restriction $x + y = 6$, the quantity $xy^2 = x(6 - x)^2$. We can therefore try to find a real number x which makes the quantity $x(6 - x)^2$ as large as possible. The idea here is to consider the function $f(x) = x(6 - x)^2$ and hope to find **a local maximum**. We can write $f(x) = 36x - 12x^2 + x^3$, and so $f'(x) = 36 - 24x + 3x^2 = 3(x^2 - 8x + 12) = 3(x-2)(x-6)$. Hence $x = 2$ and $x = 6$ are stationary points. Next, note that $f'(x) = 6x - 24$. Hence $f''(2) = -12$ and $f''(6) = 12$. It follows that the function $f(x)$ has a local maximum at the point $x = 2$. Then $y = 6 - x = 4$, with $f(2) = 32$. This choice of x and y makes xy^2 as large as possible, with value $f(2) = 32$.

Example 2: We have 20 metres of fencing material, and wish to find the largest rectangular area that we can enclose. Suppose that the rectangular area has sides x and y in metres. Then the area is equal to xy, while the perimeter is equal to $2x+2y$. Hence we wish to maximize the quantity xy subject to the restriction $2x+2y = 20$. Under the restriction $2x+2y = 20$, the quantity $xy = x(10-x)$. We can therefore try to find a real number x which makes the quantity $x(10 - x)$ as large as possible. Consider the function $f(x) = x(10 - x) = 10x - x^2$. Then $f'(x) = 10 - 2x$, and so $x = 5$ is a stationary point. Since $f''(x) = -2$, the point $x = 5$ is a local maximum. Then $y = 10 - x = 5$, with $f(5) = 25$. This choice of x and y makes xy as large as possible, with area 25 square metres.

Example 3: A manufacturer wishes to maximize the volume of cylindrical metal cans made out of a fixed quantity of metal. To understand this problem, suppose that a typical can has radius r and height *h* as shown in the picture below:



Then the total surface area is equal to $2\pi r^2 + 2\pi rh = S$, where S is fixed, so that

$$h = \frac{S}{2\pi r} - r.$$

On the other hand, the volume of such a can is equal to $V = \pi r2h$. Under the restriction (1), we ha

$$V = \pi r^2 h = \frac{Sr}{2} - \pi r^3.$$

Consider now the function

$$V(r) = \frac{Sr}{2} - \pi r^3.$$

Differentiating, we have

$$V'(r) = \frac{S}{2} - 3\pi r^2,$$

so that $r = \sqrt{S/6\pi}$ is the only stationary point, since negative values of r are meaningless. Furthermore,we have $V''(r) = -6\pi r$, and so this stationary point is a local maximum. For this value of r, we have

$$h = \frac{S}{2\pi r} - r = \sqrt{\frac{3S}{2\pi}} - \sqrt{\frac{S}{6\pi}} = \sqrt{\frac{9S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 3\sqrt{\frac{S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 2\sqrt{\frac{S}{6\pi}} = 2r.$$

This means that the most economical shape of a cylindrical can is when the height is twice the radius.

ACTIVITES FOR CHAPTER IX

ACTIVITY IX.1: For each of the following functions, find all of the stationary points. For each such stationary point, determine whether it is a local maximum, a local minimum or another type of stationary point:

a) $f(x) = 3x^2 + 6x + 9$ b) $f(x) = 6x - x^2$ c) $f(x) = 2 - 3x - 3x^2$

d) $f(x) = 6 + 9x - 3x^2 - x^3$ e) $f(x) = x + \dfrac{4}{x+1}$ f) $f(x) = 4x - 1 + \dfrac{36}{x-1}$

g) $f(x) = (x+1)^2 - (x-1)^2$ h) $f(x) = 6 - \dfrac{2}{x} - x^2$

ACTIVITY IX.2: A bullet is shot upwards at time $t = 0$ from the top of a building 176 metres tall, with an initial speed of 160 metres per second. The height of the bullet is given by $h(t) = -16t^2 + 160t + 176$ after t seconds. At what time is the bullet at maximum height above the ground? What is this height?

ACTIVITY IX.3: A rectangular beam, of width w and depth d, is cut from a circular log of diameter $a = 25$ centimetres. The beam has strength S given by $S = 2wd^2$. Find the dimension that will give the strongest beam.

[Hint: Use $d^2 + w^2 = a^2$ to relate the variables d and w.]

CHAPTER X: INTRODUCTION TO INTEGRATION

X.1. ANTIDERIVATIVES

In this chapter, we discuss the inverse process of differentiation. In other words, given a function *f*(*x*), we wish to find a function *F*(*x*) such that *F'*(*x*) = *f*(*x*). Any such function *F*(*x*) is called an **antiderivative, or indefinite integral**, of the function *f*(*x*), and we write : $F(x) = \int f(x)dx$

A first observation is that the antiderivative, if it exists, is not unique. Suppose that the function *F*(*x*) is an antiderivative of the function *f*(*x*), so that *F'*(*x*) = *f*(*x*). Let *G*(*x*) = *F*(*x*) + *C*, where *C* is any fixed real number. Then it is easy to see that *G'*(*x*) = *F'*(*x*) = *f*(*x*), so that *G*(*x*) is also an antiderivative of *f*(*x*). A second observation, somewhat less obvious, is that for any given function *f*(*x*), any two distinct antiderivatives of *f*(*x*) must differ only by a constant. In other words, **if *F*(*x*) and *G*(*x*) are both antiderivatives of *f*(*x*), then *F*(*x*) − *G*(*x*) is a constant**. In this chapter, we shall denote any such constant by *C*, with or without subscripts.An immediate consequence of this second observation is the following simple result related to the derivatives of constants already previously seen.

ANTIDERIVATIVES OF ZERO:

We have $\int 0\ dx$=C

In other words, the antiderivatives of the zero function are precisely all the constant functions. Indeed, many antiderivatives can be obtained simply by referring to various rules concerning derivatives. We list here a number of such results. The first of these is related to the already seen constant multiple rule for differentiation.

CONSTANT MULTIPLE RULE.

Suppose that a function f(*x*) *has antiderivatives. Then for any fixed real number c, we have*

$$\int cf(x)\ dx = c\int f(x)\ dx$$

ANTIDERIVATIVES OF POWERS.

a)Suppose that n is a fixed real number such that n ≠−1*. Then*

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$$

b) We have:

$$\int x^{-1} dx = \log|x| + C.$$

Proof. Part (a) is a consequence of the rule concerning derivatives of powers already seen. If *x* > 0, then part (b) is a consequence of the rule concerning the derivative of the logarithmic function. If *x* < 0, we can write |*x*| = *u*, where *u* = −*x*. It then follows from the Chain rule that

$$\frac{d}{dx}\log(|x|) = \frac{du}{dx}x\frac{dx}{du}log(u) = -\frac{1}{u} = \frac{1}{x}$$

SUM RULE:

Suppose that functions f(x) and g(x) have antiderivatives. Then

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)\,dx$$

ANTIDERIVATIVES OF TRIGONOMETRIC FUNCTIONS:

(a) We have

$$\int \cos x\,\mathrm{d}x = \sin x + C \qquad \text{and} \qquad \int \sin x\,\mathrm{d}x = -\cos x + C.$$

(b) We have

$$\int \sec^2 x\,\mathrm{d}x = \tan x + C \qquad \text{and} \qquad \int \csc^2 x\,\mathrm{d}x = -\cot x + C.$$

(c) We have

$$\int \tan x \sec x\,\mathrm{d}x = \sec x + C \qquad \text{and} \qquad \int \cot x \csc x\,\mathrm{d}x = -\csc x + C.$$

(d) We have

$$\int \sec x\,\mathrm{d}x = \log|\tan x + \sec x| + C \qquad \text{and} \qquad \int \csc x\,\mathrm{d}x = -\log|\cot x + \csc x| + C.$$

Proof. Parts **(a)–(c)** follow immediately from the rules concerning derivatives of the already seen trigonometric functions. Part (d) follows from also the already given examples.
Corresponding to the rule concerning the derivative of the exponential function, we have the following.

ANTIDERIVATIVES OF THE EXPONENTIAL FUNCTION:

We have

$$\int e^x dx = e^x + \mathrm{c}$$

Example 1: Using the sum rule, the constant multiple rule and the rule concerning antiderivatives of powers, we have

$$\int (x^2 + 3x + 1)\,\mathrm{d}x = \int x^2\,\mathrm{d}x + 3\int x\,\mathrm{d}x + \int x^0\,\mathrm{d}x = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + C.$$

Example 2: Using the sum rule and the rules concerning antiderivatives of powers and of trigonometric functions, we have

$$\int (x^3 + \sin x)\,\mathrm{d}x = \int x^3\,\mathrm{d}x + \int \sin x\,\mathrm{d}x = \frac{1}{4}x^4 - \cos x + C.$$

Example 3: We have

$$\int (\sin x + \sec x)\,\mathrm{d}x = \int \sin x\,\mathrm{d}x + \int \sec x\,\mathrm{d}x = -\cos x + \log|\tan x + \sec x| + C.$$

Example 4:

$$\int (e^x + 3\cos x)\,dx = \int e^x\,dx + 3\int \cos x\,dx = e^x + 3\sin x + C.$$

Example 5: To find

$$\int \frac{1-\sin x}{1+\sin x}\,dx,$$

note first of all that

$$\frac{1-\sin x}{1+\sin x} = \frac{(1-\sin x)(1-\sin x)}{(1+\sin x)(1-\sin x)} = \frac{1-2\sin x+\sin^2 x}{1-\sin^2 x} = \frac{1-2\sin x+\sin^2 x}{\cos^2 x}$$
$$= \sec^2 x - 2\tan x\sec x + \tan^2 x = 2\sec^2 x - 2\tan x\sec x - 1.$$

It follows that

$$\int \frac{1-\sin x}{1+\sin x}\,dx = 2\int \sec^2 x\,dx - 2\int \tan x\sec x\,dx - \int dx = 2\tan x - 2\sec x - x + C.$$

X.2: INTEGRATION BY SUBSTITUTION

We now discuss how we can use the chain rule in differentiation to help solve problems in integration. This technique is usually called integration by substitution. As we shall not prove any result here, our discussion will be only heuristic. We emphasize that the technique does not always work. First of all, we have little or no knowledge of the antiderivatives of many functions. Secondly, there is no simple routine that we can describe to help us find a suitable substitution even in the cases where the technique works. On the other hand, when the technique does work, there may well be more than one suitable substitution!

★ ***Version 1***:*If we make a substitution* $x = g(u)$, *then* $dx = g'(u)\,du$, *and*

$$\int f(x)\,dx = \int f(g(u))g'(u)\,du.$$

Example 1: Consider the indefinite integral

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

If we make a substitution $x = \sin u$, then $dx = \cos u\,du$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,du = \int du = u + C = \sin^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cos v$, then $dx = -\sin v\,dv$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\int \frac{\sin v}{\sqrt{1-\cos^2 v}}\,dv = -\int dv = -v + C = -\cos^{-1} x + C.$$

Example 2: Consider the indefinite integral

$$\int \frac{1}{1+x^2}\,dx$$

If we make a substitution $x = \tan u$, then $dx = \sec^2 u\,du$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u = \int \mathrm{d}u = u + C = \tan^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cot v$, then $\mathrm{d}x = -\csc^2 v\,\mathrm{d}v$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = -\int \frac{\csc^2 v}{1+\cot^2 v}\,\mathrm{d}v = -\int \mathrm{d}v = -v + C = -\cot^{-1} x + C.$$

Example 3: Consider the indefinite integral

$$\int x\sqrt{x+1}\,\mathrm{d}x.$$

If we make a substitution $x = u^2 - 1$, then $\mathrm{d}x = 2u\,\mathrm{d}u$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int 2(u^2-1)u^2\,\mathrm{d}u = 2\int u^4\,\mathrm{d}u - 2\int u^2\,\mathrm{d}u \\ &= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

On the other hand, if we make a substitution $x = v - 1$, then $\mathrm{d}x = \mathrm{d}v$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int (v-1)v^{1/2}\,\mathrm{d}v = \int v^{3/2}\,\mathrm{d}v - \int v^{1/2}\,\mathrm{d}v \\ &= \frac{2}{5}v^{5/2} - \frac{2}{3}v^{3/2} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

We can confirm that the indefinite integral is correct by checking that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C\right) = x\sqrt{x+1}.$$

- ★ ***Version 2:*** *Suppose that a function* $f(x)$ *can be written in the form* $f(x) = g(h(x))h'(x)$*. If we make a substitution* $u = h(x)$*, then* $\mathrm{d}u = h'(x)\,\mathrm{d}x$*, and*

$$\int f(x)\,\mathrm{d}x = \int g(h(x))h'(x)\,\mathrm{d}x = \int g(u)\,\mathrm{d}u$$

In Version 1, the variable x is initially written as a function of the new variable u, whereas **in Version 2,** the new variable u is written as a function of x. The difference, however, is minimal, as the substitution $x = g(u)$ in Version 1 has to be invertible to enable us to return from the new variable u to the original variable x at the end of the process.

Example 4: Consider the indefinite integral

$$\int x(x^2+3)^4\,\mathrm{d}x.$$

Note first of all that the derivative of the function $x^2 + 3$ is equal to $2x$, so it is convenient to make the substitution $u = x^2 + 3$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$, and

$$\int x(x^2+3)^4\,\mathrm{d}x = \frac{1}{2}\int 2x(x^2+3)^4\,\mathrm{d}x = \frac{1}{2}\int u^4\,\mathrm{d}u = \frac{1}{10}u^5 + C = \frac{1}{10}(x^2+3)^5 + C.$$

Example 5: Consider the indefinite integral

$$\int \frac{1}{x \log x} \, dx.$$

Note first of all that the derivative of the function log x is equal to $1/x$, so it is convenient to make the substitution u = log x. Then du = $(1/x)$ dx, and

$$\int \frac{1}{x \log x} \, dx = \int \frac{1}{u} \, du = \log |u| + C = \log |\log x| + C.$$

Example 6: Consider the indefinite integral

$$\int x^2 e^{x^3} \, dx.$$

Note first of all that the derivative of the function x^3 is equal to $3x^2$, so it is convenient to make the substitution $u = x^3$. Then du = $3x^2$ dx, and

$$\int x^2 e^{x^3} \, dx = \frac{1}{3} \int 3x^2 e^{x^3} \, dx = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C.$$

A somewhat more complicated alternative is to note that the derivative of the function e^{x3} is equal to $3x2\ e^{x3}$ so it is convenient to make the substitution $v = e^{x3}$. Then dv = $3x^2 e^{x3}$ dx, and

$$\int x^2 e^{x^3} \, dx = \frac{1}{3} \int 3x^2 e^{x^3} \, dx = \frac{1}{3} \int dv = \frac{1}{3} v + C = \frac{1}{3} e^{x^3} + C.$$

Example 7: Consider the indefinite integral

$$\int \tan^3 x \sec^2 x \, dx.$$

Note first of all that the derivative of the function tan x is equal to $\sec^2 x$, so it is convenient to make the substitution u = tan x. Then du = $\sec^2 x$ dx, and

$$\int \tan^3 x \sec^2 x \, dx = \int u^3 \, du = \frac{1}{4} u^4 + C = \frac{1}{4} \tan^4 x + C.$$

Occasionally, the possibility of substitution may not be immediately obvious, and a certain amount of trial and error does occur. The fact that one substitution does not appear to work does not mean that the method fails. It may very well be the case that we have used a bad substitution. Or perhaps we may slightly modify the problem first. We illustrate this point by looking at the two following examples.

Example 8: Consider the indefinite integral

$$\int \tan x \, dx.$$

Here it does not appear that any substitution will work. However, if we write

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx,$$

then we observe that the derivative of the function cos x is equal to −sin x, so it is convenient to make the substitution $u = \cos x$. Then $du = -\sin x\, dx$, and

$$\int \tan x \,\mathrm{d}x = -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{1}{u}\,\mathrm{d}u = -\log|u| + C = -\log|\cos x| + C.$$

Example 9: The indefinite integral

$$\int \frac{9+6x+2x^2+x^3}{4+x^2}\,\mathrm{d}x \qquad (1)$$

is rather daunting at first sight, but we have enough technique to study it. Note first of all that

$$\begin{aligned}9+6x+2x^2+x^3 &= 9+2x+2x^2+4x+x^3 = 9+2x+2x^2+x(4+x^2)\\ &= 1+2x+8+2x^2+x(4+x^2) = 1+2x+2(4+x^2)+x(4+x^2).\end{aligned}$$

It follows that

$$\int \frac{9+6x+2x^2+x^3}{4+x^2}\,\mathrm{d}x = \int \frac{1}{4+x^2}\,\mathrm{d}x + \int \frac{2x}{4+x^2}\,\mathrm{d}x + \int (2+x)\,\mathrm{d}x. \qquad (2)$$

To study the first integral on the right hand side of (2), we can make a substitution $x = 2\tan u$. Then $dx = 2\sec^2 u\, du$, and

$$\int \frac{1}{4+x^2}\,\mathrm{d}x = \int \frac{2\sec^2 u}{4+4\tan^2 u}\,\mathrm{d}u = \frac{1}{2}\int \mathrm{d}u = \frac{1}{2}u + C_1 = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C_1 \qquad (3)$$

To study the second integral on the right hand side of (2), we note that the derivative of the function $4 + x^2$ is equal to $2x$. If we make a substitution $v = 4+x^2$, then $dv = 2x\, dx$, and

$$\int \frac{2x}{4+x^2}\,\mathrm{d}x = \int \frac{1}{v}\,\mathrm{d}v = \log|v| + C_2 = \log(4+x^2) + C_2. \qquad (4)$$

The third integral on the right hand side of (**2**) is easy to evaluate. We have

$$\int (2+x)\,\mathrm{d}x = 2x + \frac{1}{2}x^2 + C_3. \qquad (5)$$

Substituting (3)–(5) into (2) and writing $C = C_1 + C_2 + C_3$, we obtain

$$\int \frac{9+6x+2x^2+x^3}{4+x^2}\,\mathrm{d}x = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \log(4+x^2) + 2x + \frac{1}{2}x^2 + C.$$

It may be worth checking that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \log(4+x^2) + 2x + \frac{1}{2}x^2 + C\right) = \frac{9+6x+2x^2+x^3}{4+x^2}.$$

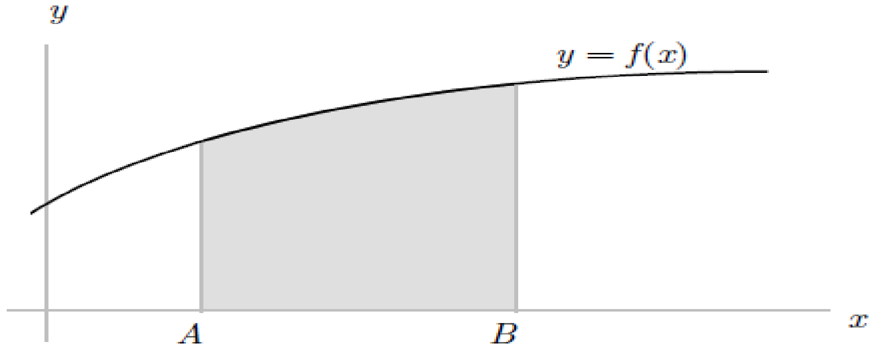
X.3: DEFINITE INTEGRALS

Suppose that $f(x)$ is a real valued function, defined on an interval $[A,B] = \{x \in \mathrm{R} : A \leq x \leq B\}$.
We shall suppose also that $f(x)$ has an antiderivative $F(x)$ for every $x \in [A,B]$.Consider that $f(x) \geq 0$ for every $x \in [A,B]$.

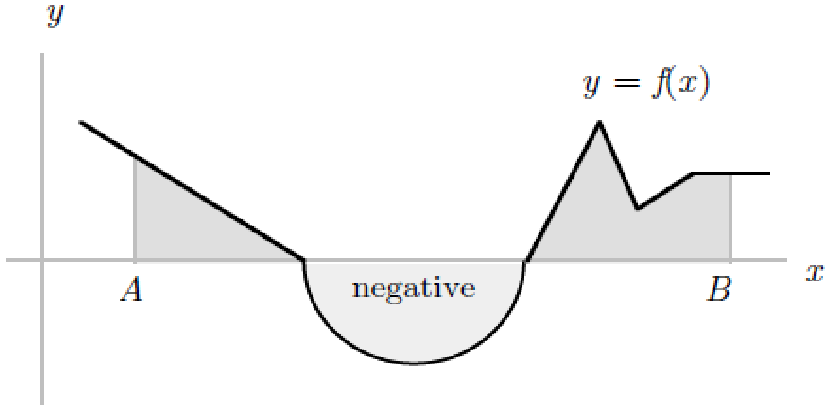
By the definite integral

$$\int_A^B f(x)\,\mathrm{d}x,$$

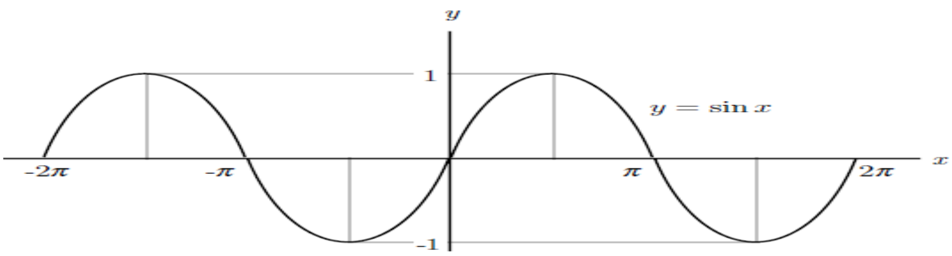
we mean the area below the curve $y = f(x)$ and above the horizontal axis $y = 0$, bounded between the vertical lines $x = A$ and $x = B$, as shown in the picture below.

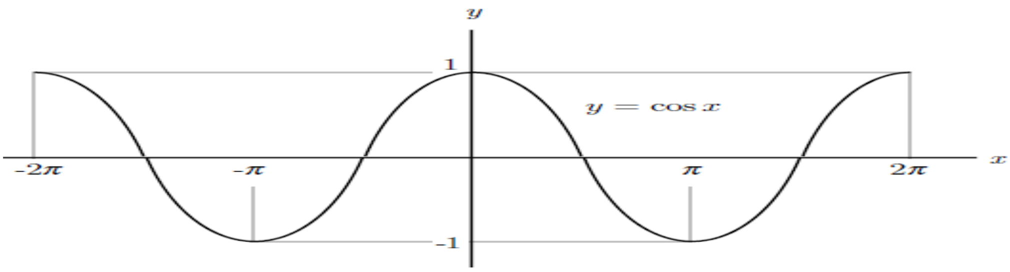


In general, we take the area between the curve $y = f(x)$ and the horizontal axis $y = 0$, bounded between the vertical lines $x = A$ and $x = B$, with the convention that the area below the horizontal axis $y = 0$ is taken to be negative, as shown in the picture below.



Example 1: If we examine the graph of the trigonometric functions **sinus x(a) and cosinus x(b) below**:





then it is easy to see that

$$\int_0^{2\pi} \sin x \, dx = 0 \qquad \text{and} \qquad \int_0^{\pi} \cos x \, dx = 0.$$

In each case, it is easy to see that the area in question above the horizontal axis $y = 0$ is equal to the area in question below this axis.

Example 2:It is easy to see that the area between the line $y = x$ and the horizontal axis $y = 0$, bounded between the vertical lines $x = 0$ and $x = 1$, is the area of a triangle with base 1 and height 1. Hence

$$\int_0^1 x\,\mathrm{d}x = \frac{1}{2}.$$

In many instances, we do not have such geometric information to help us calculate the area in question. Instead, we can use the indefinite integral.

X.3.1:FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS.

Suppose that a function F(x) satisfies F'(x) = f(x) for every x ∈ [A,B]. Then

$$\int_A^B f(x)\,\mathrm{d}x = \Big[F(x)\Big]_A^B = F(B) - F(A).$$

A simple consequence of the above is that the constant multiple rule and sum rule for indefinite integrals extend to definite integrals. For any fixed real number *c*, we have

$$\int_A^B cf(x)\,\mathrm{d}x = c\int_A^B f(x)\,\mathrm{d}x.$$

We also have

$$\int_A^B (f(x) + g(x))\,\mathrm{d}x = \int_A^B f(x)\,\mathrm{d}x + \int_A^B g(x)\,\mathrm{d}x.$$

A further consequence of the Fundamental theorem of integral calculus is a rule concerning splitting up an interval [*A*,*B*] into two. Suppose that $A < A* < B$. Then

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^{A^*} f(x)\,\mathrm{d}x + \int_{A^*}^B f(x)\,\mathrm{d}x.$$

Example 3: Returning to Example 1, we have

$$\int_0^{2\pi} \sin x\,\mathrm{d}x = \Big[-\cos x\Big]_0^{2\pi} = -\cos 2\pi + \cos 0 = 0$$

Example 4: Returning to Example 2, we have

$$\int_0^1 x\,\mathrm{d}x = \left[\frac{1}{2}x\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example 5: We have

$$\int_0^{\pi} \sin x\,\mathrm{d}x = \Big[-\cos x\Big]_0^{\pi} = -\cos\pi + \cos 0 = 2.$$

Example 6: We have

$$\int_1^2 \frac{1}{x}\,\mathrm{d}x = \Big[\log|x|\Big]_1^2 = \log 2 - \log 1 = \log 2.$$

Example 7: We have

$$\int_{-1}^{1}(x^3+x^2)\,\mathrm{d}x=\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1}=\left(\frac{1}{4}+\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)=\frac{2}{3}.$$

Example 8:. Recall Example1: Since

$$\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\sin^{-1}x+C, \qquad (6)$$

we have

$$\int_{0}^{1/2}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\left[\sin^{-1}x\right]_{0}^{1/2}=\sin^{-1}\frac{1}{2}-\sin^{-1}0=\frac{\pi}{6}.$$

To obtain (6), recall that we can use the substitution $x = \sin u$ to show that

$$\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\int\mathrm{d}u=u+C,$$

followed by an inverse substitution $u = \sin^{-1} x$. Here, we need to make the extra step of substituting the values $x = 0$ and $x = 1/2$ to the indefinite integral $\sin^{-1} x$. Observe, however, that with the substitution $x = \sin u$, the variable x increases from 0 to 1/2 as the variable u increases from 0 to $\pi/6$. But then

$$\int_{0}^{\pi/6}\mathrm{d}u=\left[u\right]_{0}^{\pi/6}=\frac{\pi}{6}=\int_{0}^{1/2}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x,$$

so it appears that we do not need the inverse substitution $u = \sin^{-1} x$. Perhaps we can directly substitute $u = 0$ and $u = \pi/6$ to the indefinite integral u.

X.3.2: DEFINITE INTEGRAL BY SUBSTITUTION – VERSION 1.

Suppose that a substitution $x = g(u)$ satisfies the following conditions:

(a) There exist $\alpha, \beta \in \mathrm{R}$ such that $g(\alpha) = A$ and $g(\beta) = B$.

(b) The derivative $g'(u) > 0$ for every u satisfying $\alpha < u < \beta$. ***Then*** $\mathrm{d}x = g'(u)\,\mathrm{d}u$*, and*

$$\int_{A}^{B}f(x)\,\mathrm{d}x=\int_{\alpha}^{\beta}f(g(u))g'(u)\,\mathrm{d}u.$$

If condition (b) above is replaced by the condition that the derivative $g\_(u) < 0$ for every u satisfying $\beta < u < \alpha$, then the same conclusion holds if we adopt the convention that

$$\int_{\alpha}^{\beta}f(g(u))g'(u)\,\mathrm{d}u=-\int_{\beta}^{\alpha}f(g(u))g'(u)\,\mathrm{d}u.$$

Example 9: To calculate the definite integral

$$\int_{0}^{1}\frac{1}{1+x^2}\,\mathrm{d}x,$$

we can use the substitution $x = \tan u$, so that $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$. Note that $\tan 0 = 0$ and $\tan(\pi/4) = 1$, and that $\sec^2 u > 0$ whenever $0 < u < \pi/4$. It follows that

$$\int_{0}^{1}\frac{1}{1+x^2}\,\mathrm{d}x=\int_{0}^{\pi/4}\frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u=\int_{0}^{\pi/4}\mathrm{d}u=\left[u\right]_{0}^{\pi/4}=\frac{\pi}{4}-0=\frac{\pi}{4}.$$

Example 10: To calculate the definite integral

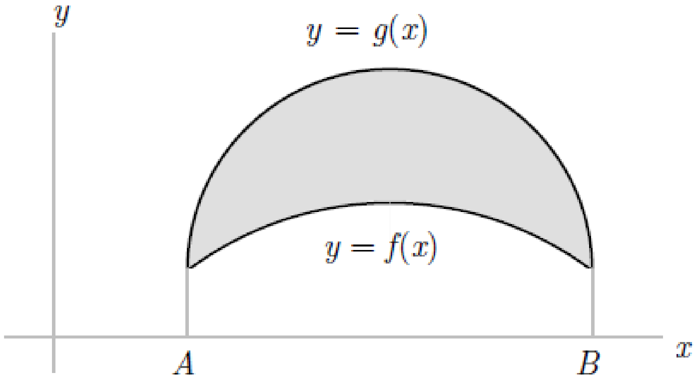
$$\int_2^4 \frac{1}{x \log x} \, \mathrm{d}x,$$

we can use the substitution $u = h(x) = \log x$, so that $\mathrm{d}u = h'(x)\,\mathrm{d}x$, where $h'(x) = 1/x > 0$ whenever $2 < x < 4$. Note also that $h(2) = \log 2$ and $h(4) = \log 4$. It follows that

$$\int_2^4 \frac{1}{x \log x} \, \mathrm{d}x = \int_{\log 2}^{\log 4} \frac{1}{u} \, \mathrm{d}u = \Big[\log |u| \Big]_{\log 2}^{\log 4} = \log\log 4 - \log\log 2 = \log\left(\frac{\log 4}{\log 2}\right) = \log 2.$$

X.4: AREAS

We conclude this chapter by describing how we may use definite integrals to evaluate areas. Suppose that the boundary of a region on the xy-plane can be described by a top edge $y = g(x)$ and a bottom edge $y = f(x)$ bounded between two vertical lines $x = A$ and $x = B$, as shown in the picture below.



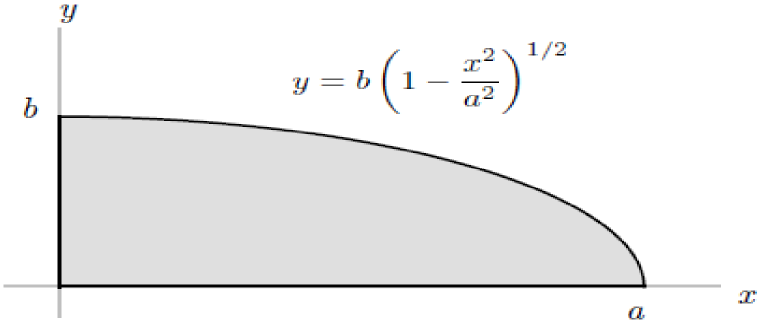
Then the area of the region is given by the definite integral

$$\int_A^B (g(x) - f(x)) \, \mathrm{d}x.$$

Example 1: We wish to show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b \in \mathrm{R}$ are positive, is equal to πab. To do this, we may consider the quarter of the ellipse in the first quadrant, as shown in the picture below.



It follows that the shaded region has area

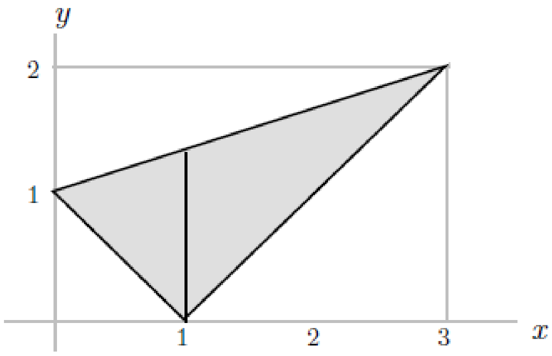
$$\int_0^a b\left(1-\frac{x^2}{a^2}\right)^{1/2} \mathrm{d}x.$$

We can use the substitution $x = g(u) = a \sin u$. Then $g(0) = 0$ and $g(\pi/2) = a$. Furthermore, we have $\mathrm{d}x = g'(u)\,\mathrm{d}u$, where $g(u) = a \cos u > 0$ whenever $0 < u < \pi/2$. It follows that

$$\int_0^a b\left(1-\frac{x^2}{a^2}\right)^{1/2} \mathrm{d}x = \int_0^{\pi/2} ab(1-\sin^2 u)^{1/2}\cos u\,\mathrm{d}u = ab\int_0^{\pi/2}\cos^2 u\,\mathrm{d}u$$

$$= ab\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2u\right)\mathrm{d}u = ab\left[\frac{1}{2}u+\frac{1}{4}\sin 2u\right]_0^{\pi/2} = \frac{\pi ab}{4}$$

Example 2: We wish to evaluate the area of the triangle with vertices (0, 1), (1, 0) and (3, 2). To do this, we split the triangle into two regions as shown in the picture below.



The triangle on the left is bounded between the vertical lines $x = 0$ and $x = 1$, and the top edge and the bottom edge are given respectively by

$$y = \frac{1}{3}x + 1 \qquad \text{and} \qquad y = 1 - x.$$

The triangle on the right is bounded between the vertical lines $x = 1$ and $x = 3$, and the top edge and the bottom edge are given respectively by

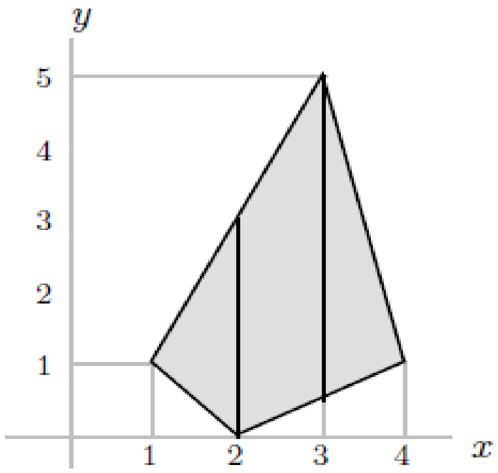
$$y = \frac{1}{3}x + 1 \qquad \text{and} \qquad y = x - 1.$$

It follows that the area of the original triangle is given by

$$\int_0^1 \left(\left(\frac{1}{3}x+1\right)-(1-x)\right)\mathrm{d}x + \int_1^3 \left(\left(\frac{1}{3}x+1\right)-(x-1)\right)\mathrm{d}x$$

$$= \int_0^1 \frac{4}{3}x\,\mathrm{d}x + \int_1^3\left(2-\frac{2}{3}x\right)\mathrm{d}x = \left[\frac{2}{3}x^2\right]_0^1 + \left[2x-\frac{1}{3}x^2\right]_1^3 = 2.$$

Example 3: We wish to evaluate the area of the quadrilateral with vertices (1, 1), (2, 0), (4, 1) and (3, 5). To do this, we split the quadrilateral into three regions as shown in the picture below**.**



The triangle on the left is bounded between the vertical lines $x = 1$ and $x = 2$, and the top edge and the bottom edge are given respectively by

$$y = 2x - 1 \qquad \text{and} \qquad y = 2 - x.$$

The quadrilateral in the middle is bounded between the vertical lines $x = 2$ and $x = 3$, and the top edge and the bottom edge are given respectively by

$$y = 2x - 1 \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

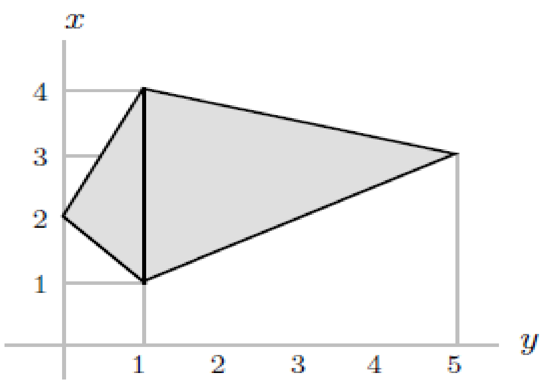
The triangle on the right is bounded between the vertical lines $x = 3$ and $x = 4$, and the top edge and the bottom edge are given respectively by

$$y = 17 - 4x \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

It follows that the area of the original quadrilateral is given by

$$\int_1^2 ((2x-1)-(2-x))\,\mathrm{d}x + \int_2^3 \left((2x-1) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x + \int_3^4 \left((17-4x) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x$$
$$= \int_1^2 (3x-3)\,\mathrm{d}x + \int_2^3 \frac{3}{2}x\,\mathrm{d}x + \int_3^4 \left(18 - \frac{9}{2}x\right)\mathrm{d}x$$
$$= \left[\frac{3}{2}x^2 - 3x\right]_1^2 + \left[\frac{3}{4}x^2\right]_2^3 + \left[18x - \frac{9}{4}x^2\right]_3^4 = \frac{15}{2}.$$

Alternatively, we can transpose the picture above and split the quadrilateral into two regions as shown in the picture below:



Note that the roles of x and y are now interchanged. The triangle on the left is bounded between the vertical lines $y = 0$ and $y = 1$, and the top edge and the bottom edge are given respectively by

$$x = 2y + 2 \qquad \text{and} \qquad x = 2 - y$$

The triangle on the right is bounded between the vertical lines $y = 1$ and $y = 5$, and the top edge and the bottom edge are given respectively by

$$x = \frac{17}{4} - \frac{1}{4}y \qquad \text{and} \qquad x = \frac{1}{2}y + \frac{1}{2}.$$

It follows that the area of the original quadrilateral is given by

$$\int_0^1 ((2y+2) - (2-y))\,\mathrm{d}y + \int_1^5 \left(\left(\frac{17}{4} - \frac{1}{4}y\right) - \left(\frac{1}{2}y + \frac{1}{2}\right)\right)\,\mathrm{d}y$$

$$= \int_0^1 3y\,\mathrm{d}y + \int_1^5 \left(\frac{15}{4} - \frac{3}{4}y\right)\,\mathrm{d}y = \left[\frac{3}{2}y^2\right]_0^1 + \left[\frac{15}{4}y - \frac{3}{8}y^2\right]_1^5 = \frac{15}{2}$$

as before.

ACTIVITIES FOR CHAPTER X

ACTIVITY X.1: Find each of the following indefinite integrals.

a) $\int \sqrt{3}\,dx$; **b)** $\int (5x+3)dx$; $c)$ $\int (2x^2-3x+1)dx$;d)

$d)$ $\int x^3 dx$; **e)** $\int (x-2)(x+3)dx$; **f)** $\int (1-2\cos x)\,dx$

ACTIVITY X.2: Evaluate each of the following indefinite integrals using the given substitution.

a) $\int \frac{x^2}{\sqrt{2+x^3}}$ dx ;**b)** $\int \sin 4x\,dx$ **c)** $\int \frac{dx}{(2x+1)^2}$;$d)$ $\int \frac{x+3}{(x^2+6x)^2)}$ dx.

Note: In **a)** use the substitution u=x^3+2**; in b)** use the substitution u=4x**; in c)** use the substitution u=2x + 1 and **in d)** use the substitution u=x^2+ 6x.

ACTIVITY X.3: Evaluate each of the following indefinite integrals.

a) $\int \cos 2x dx$; $b)$ $\int \sqrt{x-1}dx$; $c)$ $\int x^2 \cos(1-x^3)dx$; **d)** $\int x sin x^2 dx$

e) $\int \frac{1}{(1-3x)^4}$ **dx** ; $f)$ $\int \frac{x}{\sqrt{x^2+1}}\,dx$

ACTIVITY X.4: Evaluate each of the following definite integrals.

a) $\int_1^2 2x\,dx$ b) $\int_1^3 \frac{1}{x}\,dx$ c) $\int_0^2 e^{-x}\,dx$

d) $\int_2^3 (3x+1)\,dx$ e) $\int_0^\pi \sin x\,dx$ f) $\int_3^6 (x-3)^2\,dx$

g) $\int_0^2 \frac{1}{4+x^2}\,dx$ h) $\int_0^1 xe^{x^2}\,dx$ i) $\int_0^a (x^2+a^2)\,dx$

ACTIVITY X.5:

- Draw the graphs of the line *y* = *x* and the parabola *y* = x^2.
- Find the two points of intersection of the two curves.
- Use definite integrals to find the area bounded between the two curves.

ACTIVITY X.6: Find the area of the triangle with vertices (0, 0), (4, 3) and (1, 5).