(10) Let $5x^2 + 16y = 0$ to the parabolla
a) Calculate the coordinates of focus
b) Give the equation of the director
c) Give the equation of tangent which is parallel to $\Delta \equiv 2x + y - 3 = 0$

e) Give the equation of the tangent which are intercept on the $P(2,4)$

PART II
CONICS

CHAP IV QUADRATIC CURVES

1- Def: Let $P(x,y)$ be any point of the plane π which is verifying the equation.

$$\boxed{Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0}$$

A, B, C, D and F are parameters (real or complex) the equation (1) is representing the quadratic curves

→ If $A = C$ quadratic curve is a circle.
→ $A \neq C$ Ellipse or hyperbola
→ $A = 0$ $(C \neq 0)$ or $C = 0$ $(A \neq 0)$
We have the parabola.

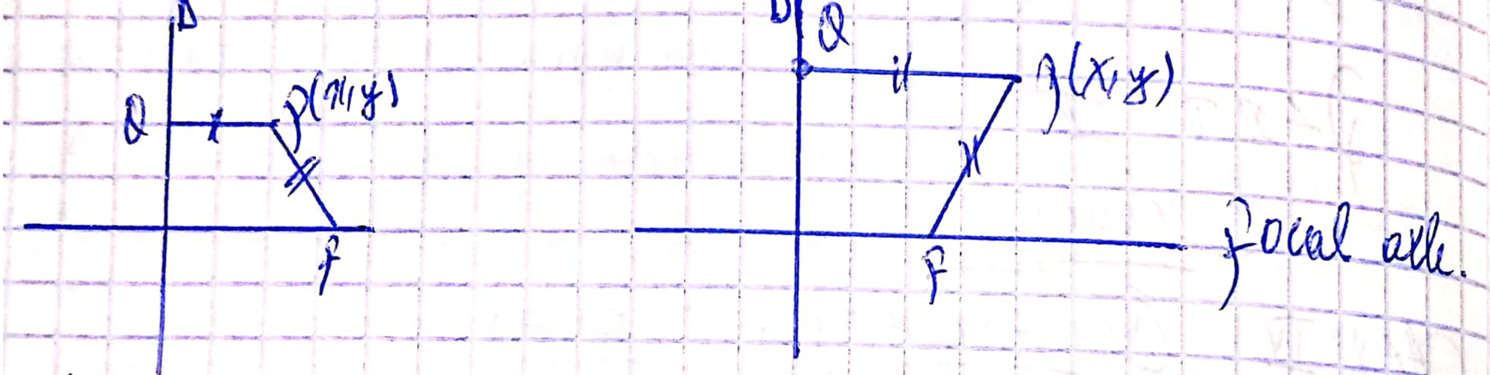
We are going to study the quadratic curve as

- Parabola
- Ellipse
- Hyperbola.

2.1. Study of parabola

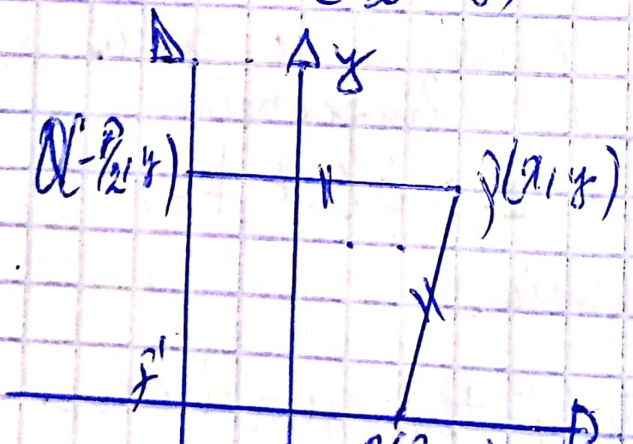
We called parabola of the focus F and of the direction straight line Δ passing through the virtual focus F and of the point $P(x,y)$ the locating on the same distance from P to F and from P to the perpendicular on Δ at the point Q.

$\|\vec{PF}\| = \|\vec{PQ}\|$ or $\overline{PF} = \overline{PQ}$



Let consider the point $P(x,y)$ of the parabola and let P be the distance between foci, the direction straight line $d \equiv x = -\frac{P}{2}$ and the focus $F(\frac{P}{2}, 0)$

The perpendicular point from P to Δ will have coordinate $Q(\frac{P}{2}, y)$ from P to Δ will have coordinate $Q(-\frac{P}{2}, y)$



$P = \|\vec{FF'}\| = \overline{FF'}$

$F(\frac{P}{2}, 0)$ and $F'(-\frac{P}{2}, 0)$

From the definition

$\|\vec{PF}\| = \|\vec{PQ}\|$ or $\overline{PF} = \overline{PQ}$

$\|\vec{PF}\| = \sqrt{(x-\frac{P}{2})^2 + (y-0)^2} = \sqrt{(x-\frac{P}{2})^2 + y^2}$

$\|\vec{PQ}\| = \sqrt{(x+\frac{P}{2})^2 + (y-y)^2} = \sqrt{(x+\frac{P}{2})^2}$

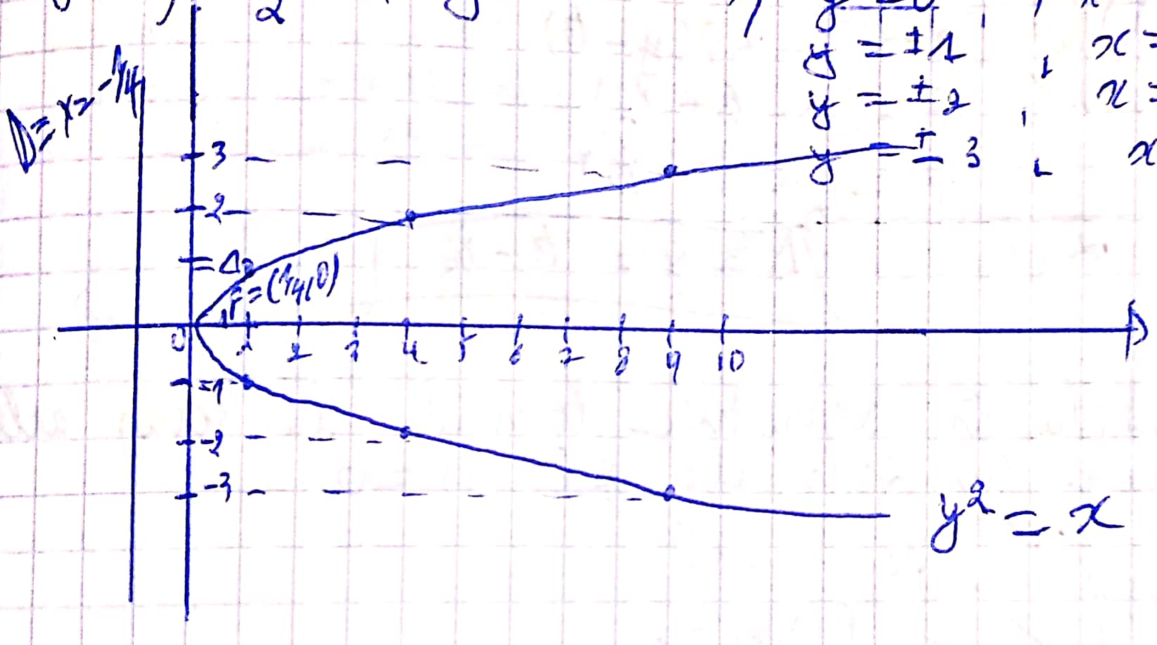
$$\|\vec{PF}\| = \|\vec{PQ}\| = \sqrt{(x-\frac{p}{2})^2 + y^2} = \sqrt{(x+\frac{p}{2})^2}$$

$$\cancel{x^2} - px + \cancel{\frac{p^2}{4}} + y^2 = \cancel{x^2} + px + \cancel{\frac{p^2}{4}}$$

$$y^2 = px + px \Rightarrow \boxed{y^2 = 2px} \text{ or } \boxed{y = \pm\sqrt{2px}}$$

p = distance between F and F'

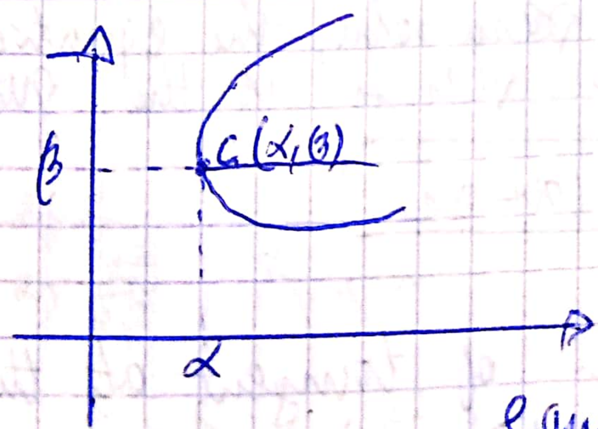
Let $p = \frac{1}{2} \Rightarrow y^2 = x$ if $y=0$, $x=0$
$y = \pm 1$, $x = 1$
$y = \pm 2$, $x = 4$
$y = \pm 3$, $x = 9$



2-2 Shifted parabola

Let consider the parabola of the centre $C(\alpha, \beta)$. This will have equation

$$\boxed{(y-\beta)^2 = 2p(x-\alpha)}$$

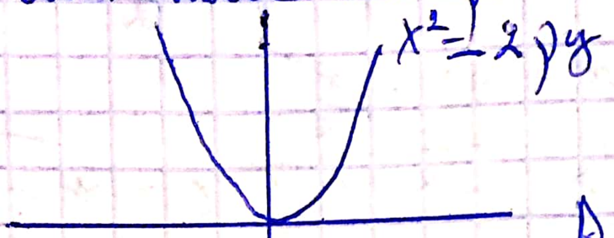


$F = (\frac{p}{2} + \alpha, \beta)$

$F' = (-\frac{p}{2} + \alpha, \beta)$

The Direction straight line will have equation $\boxed{D: x = -\frac{p}{2} + \alpha}$

We can have the parabola $x^2 = 2py$

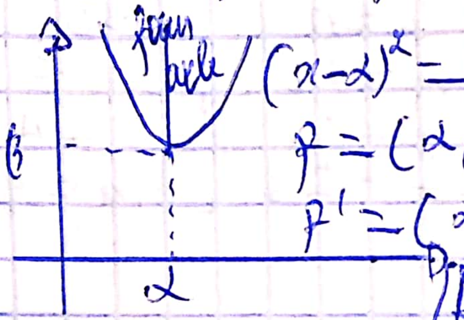


$F(0, \frac{p}{2})$

$F'(0, -\frac{p}{2})$

$\Delta \equiv y = -\frac{p}{2}$

This can be shifted to the centre $C(\alpha, \beta)$ and have equation $(x-\alpha)^2 = 2p(y-\beta)$



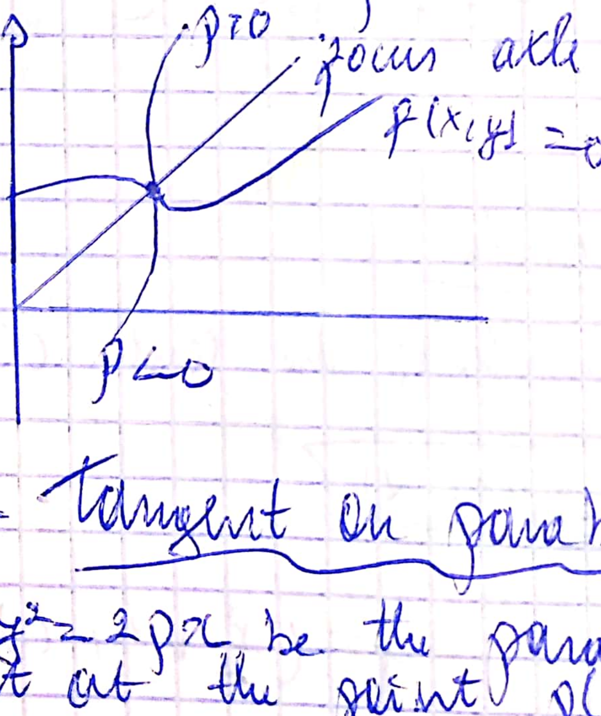
$(x-\alpha)^2 = 2p(y-\beta)$

$F = (\alpha, \beta + \frac{p}{2})$

$F' = (\alpha, \beta - \frac{p}{2})$

$\boxed{\Delta \equiv y = \beta - \frac{p}{2}}$

The parabola is symmetric toward the focus axle
we can have the parabola $f(x, y) = 0$



3- Tangent on parabola

Let $y^2 = 2px$ be the parabola, the equation of tangent at the point $p(x_1, y_1)$ at the parabola is

$$\boxed{yy_1 = p(x + x_1)}$$

Proof: The equation of tangent at the point $p(x_1, y_1)$ is $(y - y_1) = y'_p(x - x_1)$

y'_p is the slope of tangent at $p(x_1, y_1)$.

$$(y^2)' = (2px)' \Rightarrow 2y'y = 2p$$

$$y' = \frac{2p}{2y} = \frac{p}{y} \qquad \boxed{y'_p = \frac{p}{y_1}}$$

Replacing in the equation of tangent to have

$$y - y_1 = \frac{p}{y_1}(x - x_1)$$

$$y_1(y - y_1) = p(x - x_1)$$

$$y y_1 - y_1^2 = px - px_1$$ If $p(x_1, y_1)$ is the point of the parabola $y^2 = 2px$, then $y_1^2 = 2px_1$ the equation of tangent becomes

$$yy_1 - 2px_1 = px - px_1 \Rightarrow yy_1 = px - px_1 + 2px_1$$

$$yy_1 = px + px_1 \Rightarrow \boxed{yy_1 = p(x + x_1)}$$

Equation of tangent on parabola

4. Parametric Equation of the parabola

Let $y^2 = 2px$ be the parabola, the parametric equation of the parabola is $\begin{cases} x = f(t) \\ y = f'(t) \end{cases}$

$1^\circ \begin{cases} x = \dfrac{t^2}{2p} \\ y = t \end{cases}$ $\qquad 2^\circ \begin{cases} x = 2pt^2 \\ y = 2pt \end{cases}$

$y^2 = 2px$

$y^2 = 2p\dfrac{t^2}{2p}$ $\qquad \begin{cases} x = \dfrac{t^2}{2p} \\ y = t \end{cases}$

$y^2 = t^2 \Rightarrow y = t$ is the parametric equation

$y^2 = 2px$

If $x = 2pt^2$, then

$y^2 = 2p(2pt^2)$

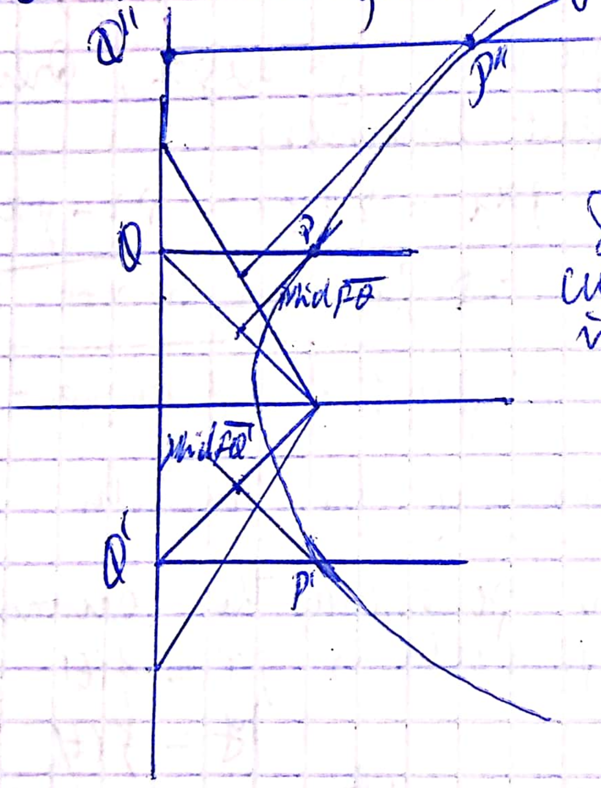
$y^2 = 4p^2t^2$

$y = 2pt$

$\begin{cases} x = 2pt^2 \\ y = 2pt \end{cases}$ is the parametric equation of the parabola $y^2 = 2px$.

5- Point of parabola

Let consider the parabola of the focus F and of the direction straight line Δ and Q be the direction straight line. The intersection of the mediator of the segment $\overline{FQ}$ and the perpendicular to Δ on Q is the point of the parabola.



N.B One set of straight lines perpendicular to the segment FQ is tangent on the parabola $y^2 = 2px$

6- Application

Verify if the following equations are parabola. Give the coordinates of centre, focus and the direction straight line

1. $y = \frac{1}{2}x^2 + x + 2$
2. $y^2 = 4 - 6x$
3. $x = 2y^2 - 12y + 14$
4. $y = 4x^2 - 8x + 7$
5. $x = -\frac{1}{4}y^2 + y$
6.

Solution.

$y = \frac{1}{2}x^2 + x + 2$

The equation of shifted parabola will be

$(y-\beta)^2 = 2p(x-\alpha)$ or $(x-\alpha)^2 = 2p(y-\beta)$

$y - 2 = \frac{1}{2}x^2 + x \Rightarrow 2(y-2) = x^2 + 2x$

$2(y-2) = (x+1)^2 - 1 \Rightarrow 2y - 4 + 1 = (x+1)^2$

$2y - 3 = (x+1)^2 \cancel{= 2(y-\frac{3}{2})}$

$(x-\alpha)^2 = 2p(y-\beta)$ $\boxed{(x+1)^2 = 2(y-\frac{3}{2})}$

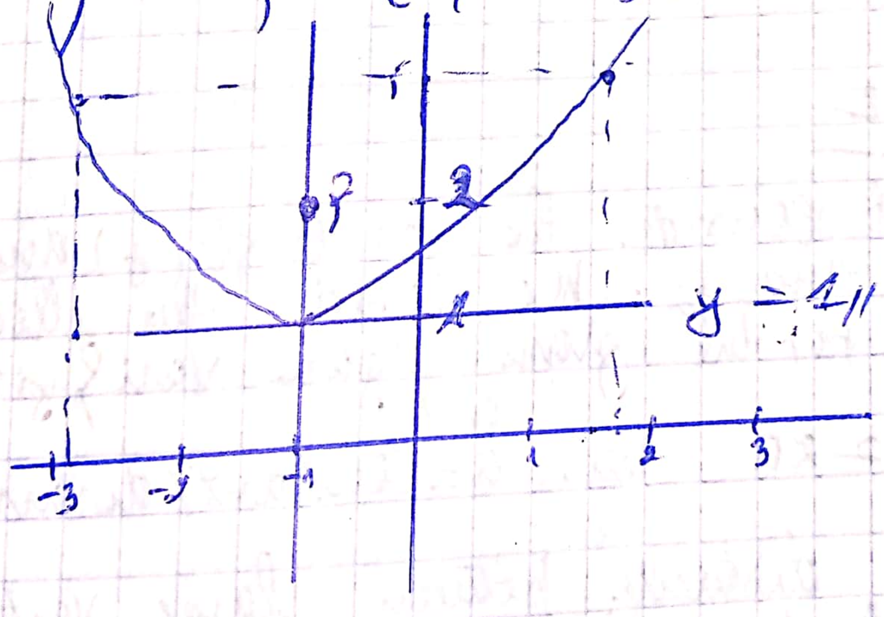
$\alpha = -1 \quad \beta = \frac{3}{2} \quad p = 1 \quad p/2 = \frac{1}{2}$

It is the parabola of the $C(-1, \frac{3}{2})$

the equation of the direction line is

$y = \beta - \frac{p}{2} = \frac{3}{2} - \frac{1}{2} = 1$

The focus $F = (\alpha, \beta + \frac{p}{2}) = (-1, 2)$.



$C(-1, \frac{3}{2})$

$D \equiv y = 1$

$F = (-1, 2)$

$y = \frac{3}{2} \Rightarrow x = -1$

$y = 5 \Rightarrow (x+1)^2 = 2(5 - \frac{3}{2}) \Rightarrow (x+1)^2 = 7$

$x = \pm\sqrt{7} - 1$

$x = 2.64 - 1 = 1.64$

$x = -2.64 - 1 = -3.64$

③ $x = 2y^2 - 12y + 14$

$x - 14 = 2y^2 - 12y$

$x - 14 = 2(y^2 - 6y)$

$\frac{x-14}{2} = y^2 - 6y$

$\frac{x-14}{2} = (y-3)^2 - 9$

$\frac{1}{2}\{x - 14\} + 9 = (y-3)^2$

$\frac{1}{2}\{x - 14 + 18\} = (y-3)^2$

$\frac{1}{2}(x+4) = (y-3)^2$

$$\boxed{(y-3)^2 = \frac{1}{2}(x+4)}$$

$(y-(3))^2 = 2p(x-\alpha)$

$C(\alpha, b) = C(-4, 3)$

$p = \frac{1}{4} \quad \frac{p}{2} = \frac{1}{8}$

$f = (\alpha + \frac{p}{2}, b) = (-4 + \frac{1}{8}, 3)$

The direction straight line

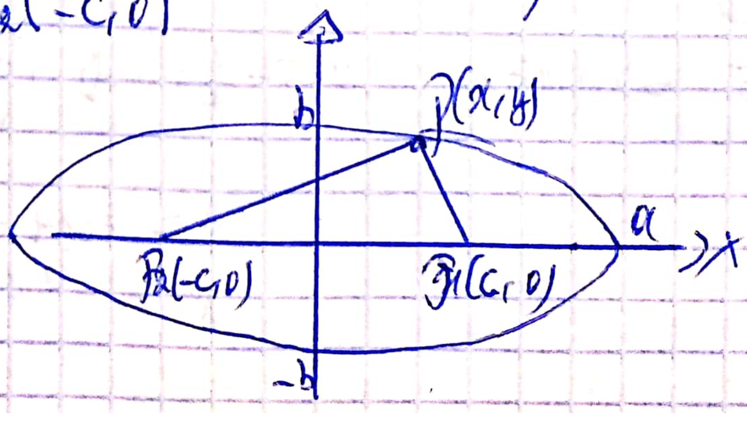
$D \equiv y = -4 - \frac{1}{8}$

3. Ellipse

1- Definition: Let consider the point $p(x, y)$ and the foci f_1 and f_2. We called the ellipse of the foci f_1 and f_2, the plane figure verifying

$\|\overrightarrow{pf_2}\| + \|\overrightarrow{pf_2}\| = 2a$ with a = constant/distance

Let $l = 2a$ be the distance between large vertices and $\|\overrightarrow{f_1f_2}\| = 2c$. Distance between foci with $f_1(c, 0)$ and $f_2(-c, 0)$



Ellipse is $\|\vec{PF_1}\| + \|\vec{PF_2}\| = 2a$

$\|\vec{PF_1}\| = \sqrt{(x-c)^2+(y-0)^2} = \sqrt{x^2-2cx+c^2+y^2}$

$\|\vec{PF_2}\| = \sqrt{(x+c)^2+(y-0)^2} = \sqrt{x^2+2cx+c^2+y^2}$

$\|\vec{PF_1}\| = 2a - \|\vec{PF_2}\|$

$\sqrt{x^2-2cx+c^2+y^2} = 2a - \sqrt{x^2+2cx+c^2+y^2}$

$\cancel{x^2} - 2cx + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\sqrt{x^2+2cx+c^2+y^2} + \cancel{x^2} + 2cx + \cancel{c^2} + \cancel{y^2}$

$-2cx - 2cx - 4a^2 = -4a\sqrt{x^2+2cx+c^2+y^2}$

$-\cancel{4}(cx+a^2) = -\cancel{4}(a\sqrt{x^2+2cx+c^2+y^2})$

$c^2x^2 + 2cxa^2 + a^4 = a^2(x^2+2cx+c^2+y^2)$

$c^2x^2 + \cancel{2cxa^2} + a^4 = a^2x^2 + \cancel{2cxa^2} + a^2c^2 + a^2y^2$

$a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$

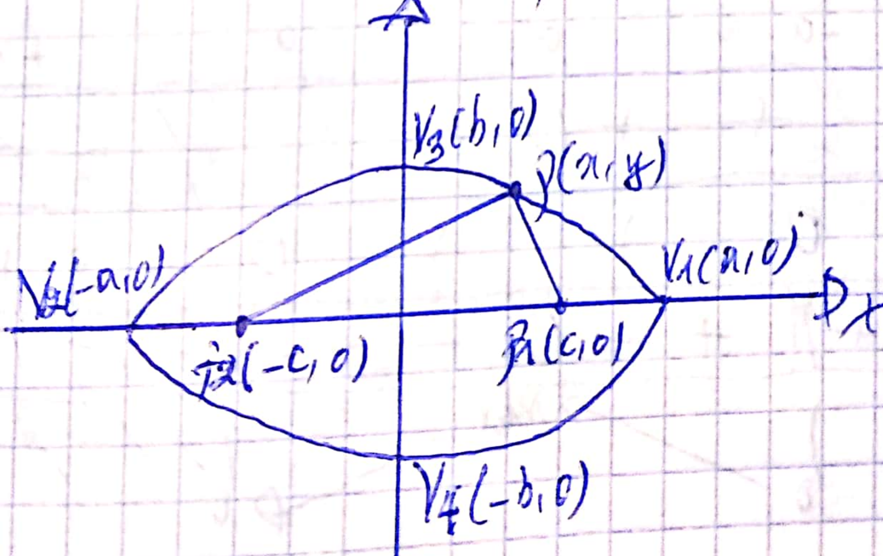
$a^2(a^2-c^2) = (a^2-c^2)x^2 + a^2y^2$ Let $b^2 = a^2 - c^2$

Then $a^2b^2 = a^2y^2 + b^2x^2$

$$1 = \frac{b^2x^2+a^2y^2}{a^2b^2}$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Carterian equation of the ellipse.



The Analysis resolution of the function $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\boxed{y = \pm \frac{b}{a}\sqrt{a^2 - x^2}}$$

1° Domain $\sqrt{a^2-x^2} \geq 0$

$a^2 - x^2 \geq 0 \Rightarrow x = \pm a$

	$-\infty$	$-a$		a	$+\infty$
a^2-x^2	$-$	0	$+ +$	0	$-$

$Dom = [-a, a]$

2° Boundaries limits

$$\lim_{x\to -a} \frac{b}{a}\sqrt{a^2-x^2} = \frac{b}{a}\sqrt{a^2-a^2} = 0$$

$$\lim_{x\to a} \frac{b}{a}\sqrt{a^2-x^2} = \frac{b}{a}\sqrt{a^2-a^2} = 0$$

3° No Asymptotes.

4° Find first derivative and variation.

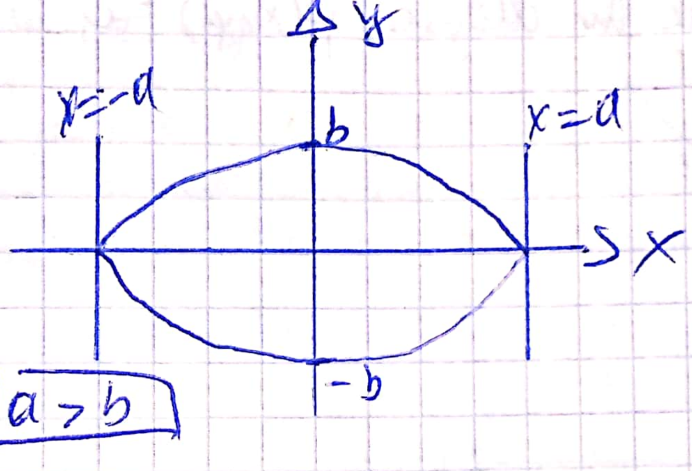
$$y' = \frac{b}{a}\left(\sqrt{a^2-x^2}\right)' = \frac{\frac{b}{a}(-2x)}{2\sqrt{a^2-x^2}} = \frac{-\frac{b}{a}x}{\sqrt{a^2-x^2}}$$

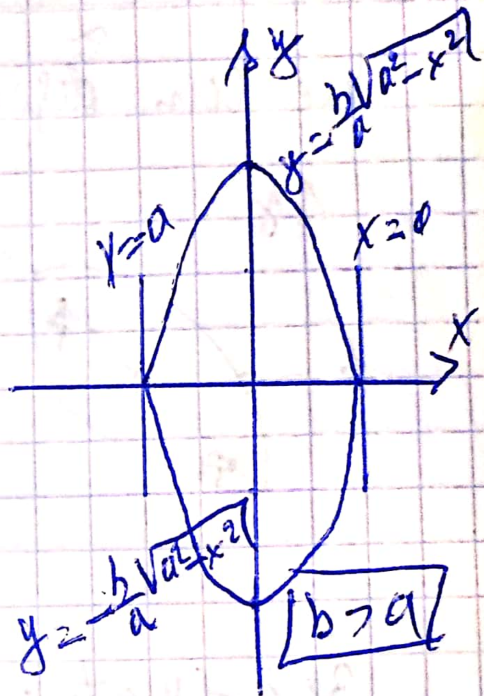
	$-\infty$	$-a$		0		a	$+\infty$
$-\frac{b}{a}x$	$+$		$+$	$+ \; 0$	$-$		$-$
$\sqrt{a^2-x^2}$	$+$	0	$+$		$+$	0	$+ + + +$
y'	$+ +$	$\|\|$	$+ +$	0	$- - -$	$\|\|$	$-$
y		0	$\nearrow$	Max	$\searrow$	0	

$y'(a) = \frac{-\frac{b}{a}a}{\sqrt{a^2-a^2}} \Rightarrow \frac{-b}{0} = -\infty$ $\quad x=+a$ is vertical tangent

$y'(-a) = \frac{-\frac{b}{a}(-a)}{\sqrt{a^2-a^2}} \Rightarrow \frac{b}{0} = +\infty$ $\quad x=-a$ is vertical tangent.

$y(0) = \frac{b}{a}\sqrt{a^2-0^2} \Rightarrow \frac{b}{a}a = b$ $\quad$ Max$(0, b)$.

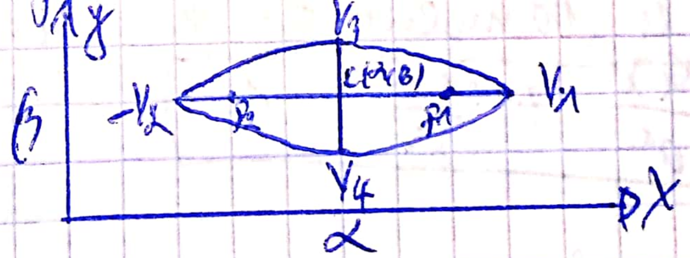




2. Shifted Ellipse

The equation of the ellipse centered on $C(\alpha, \beta)$ is

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$$



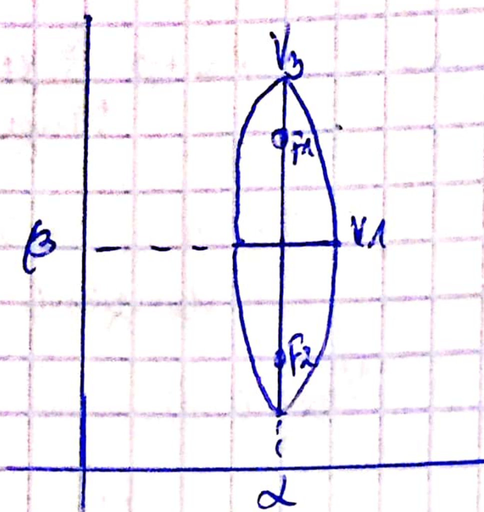
$C(\alpha, \beta)$

$\boxed{b^2 = a^2 - c^2}$

$f_1(\alpha+c, \beta)$ $\quad V_1(\alpha+a, \beta)$ $\quad V_3(\alpha, \beta+b)$

$f_2(\alpha-c, \beta)$ $\quad V_2(\alpha-a, \beta)$ $\quad V_4(\alpha, \beta-b)$

Let consider $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$ $\quad b > a$.



$a^2 = b^2 - c^2$

$F_1(\alpha, \beta + c)$

$F_2(\alpha, \beta - c)$

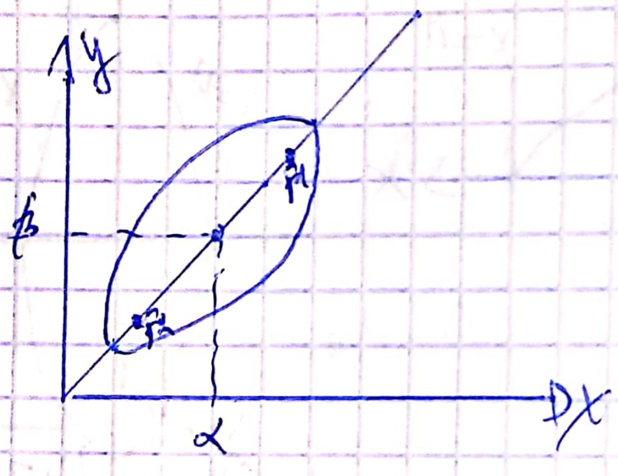
$V_1(\alpha + a, \beta)$

$V_2(\alpha - a, \beta)$

$V_3(\alpha, \beta + b)$

$V_4(\alpha, \beta - b)$

We can define the ellipse $f(x, y) = 0$



3. Equation of tangent on the ellipse

Let consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, The equation of tangent at the point $P(x_1, y_1)$ on the ellipse is

$$\boxed{\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1}$$

Proof:

The equation of tangent at $P(x_1, y_1)$ is

$y - y_1 = y'_P(x - x_1)$ where y'_P is the slope at the point $P(x_1, y_1)$

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse

The derivative both side gives:

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)'=(1)' \Rightarrow \frac{2x}{a^2}+\frac{2y'y}{b^2}=0 \Rightarrow \frac{2y'y}{b^2}=-\frac{2x}{a^2}$$

$$\boxed{y'=\frac{-b^2x}{a^2y}}$$ and $y'_p=-\frac{b^2x_1}{a^2y_1}$ at the point $p(x_1,y_1)$

Replacing $y'_p=\frac{-b^2x_1}{a^2y_1}$ into the equation of tangent

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1(y-y_1)=-b^2x_1(x-x_1)$$

$$a^2y_1y-a^2y_1^2=-b^2x_1(x-x_1)$$

$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$

$$a^2y_1y+b^2x_1x=b^2x_1^2+a_1^2y_1^2$$

$$a^2b^2\left(\frac{y_1y}{b^2}+\frac{x_1x}{a^2}\right)=a^2b^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)$$

$$\frac{y_1y}{b^2}+\frac{x_1x}{a^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

The point $p(x_1,y_1)$ is the point of the ellipse, Then

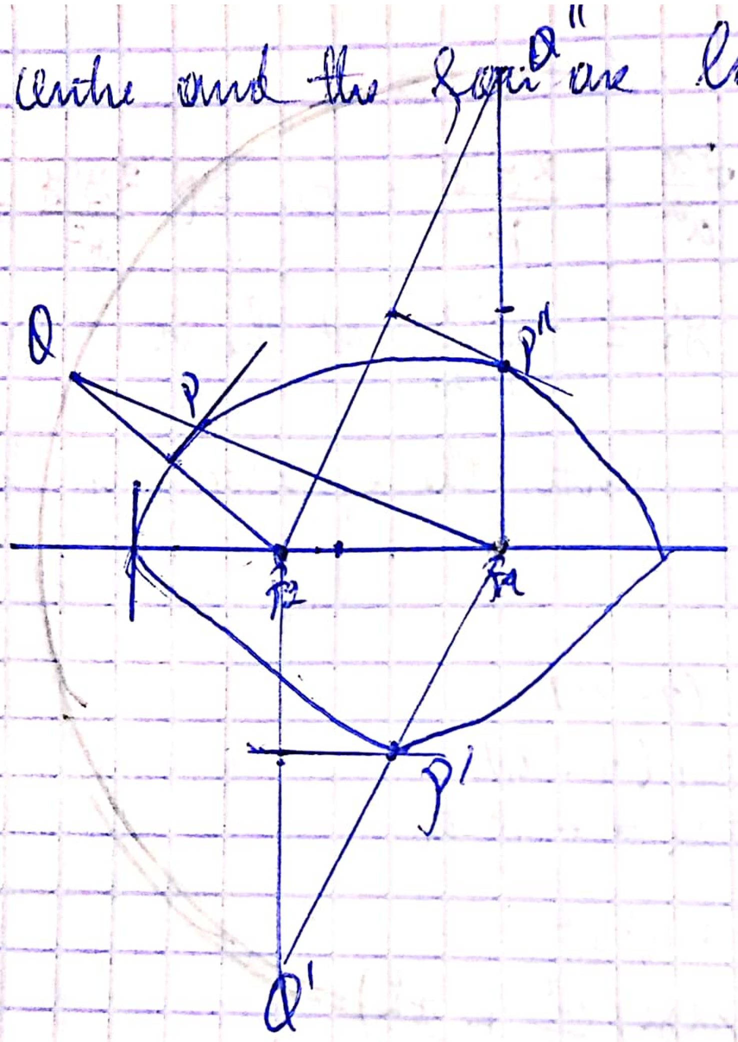
$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$, so $\boxed{\frac{y_1y}{b^2}+\frac{x_1x}{a^2}=1}$ is the equato of tangen on the ellipse.

4. Point on the ellipse.

Let consider the ellipse of the foci $f_1(c,0)$ and $f_2(-c,0)$ and of the large length $l=2a$

Let have a circle of a radius $R=2a$ where the center is on one focus f_1. Let Q be the point of the circle and f_2Q be the distance from circle to the focus f_2. The intersection between the mediator on the segment f_2Q with the segment f_1Q is the point p of the ellipse.

The centre and the foci are located on focus axle



5. Parametric Equation of Ellipse

Let consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The parametric equation of the ellipse is

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$

$$\boxed{b^2x^2 + a^2y^2 = a^2b^2}$$ divide both side by a^2b^2

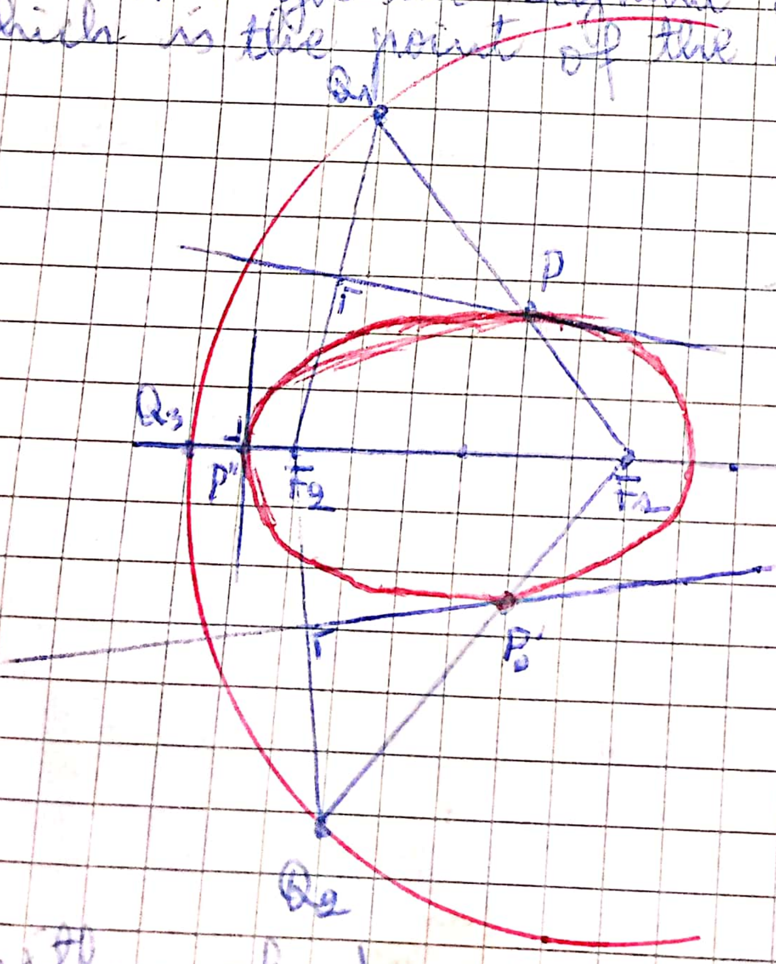
to get $\dfrac{b^2x^2 + a^2y^2}{a^2b^2} = 1$

$$\Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$ $a > c$ and $b^2 = a^2 - c^2$

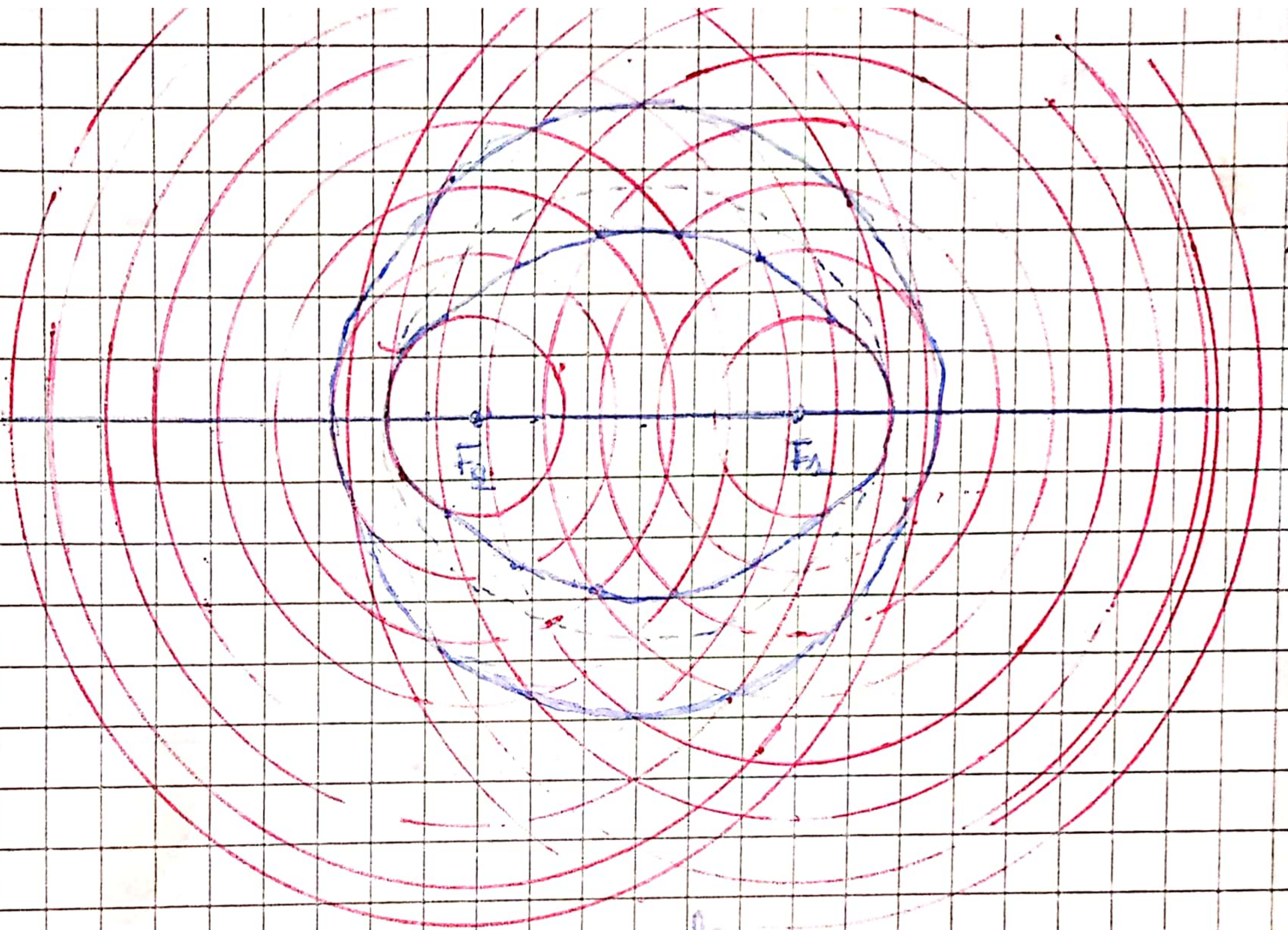
this is the equation of the ellipse which is centered on (0,0) where the axis are a and b

4 Construction of a point of the ellipse

Let consider the ellipse of the focus F_1 and F_2 of the length $l = 2a$. Let us draw the circle of the radius l and of the center F_1 (or F_2). Let us join F_1 on the point Q of the circle. Let us also find the mediator to the segment QF_2 which passes through the segment QF_1 on the point P which is the point of the circle.



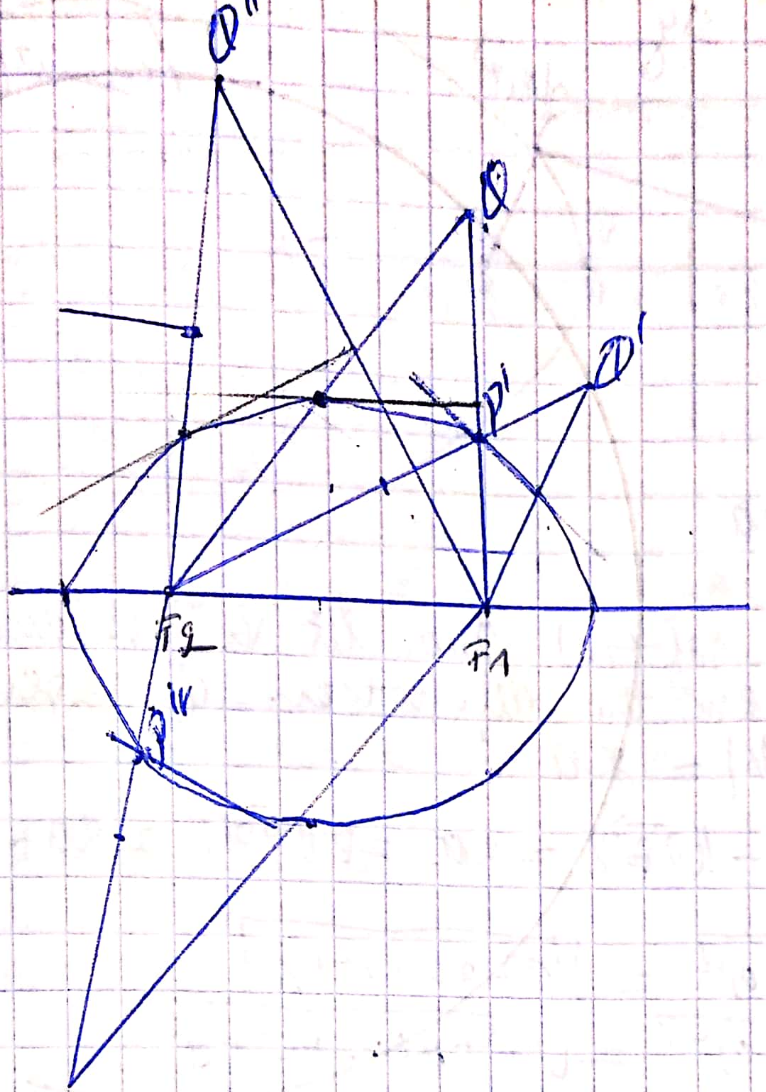
b) Algorithm of drawing an ellipse

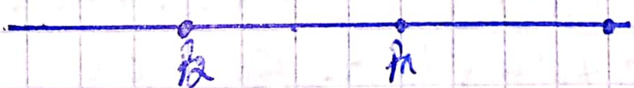


1. Let draw the focus axis where we fix F1 and F2 on that axis
2. Let draw the concentric circles on the points F1 and F2 which are equal on each focus
3. The points where the concentric circles intersect between F1 and F2 are the points of the need ellipse.
4. There is more than one ellipse, it is depend on the choice
5 - - -

I/ Graphic study of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Cfr S5)

Domf = [-a, a]



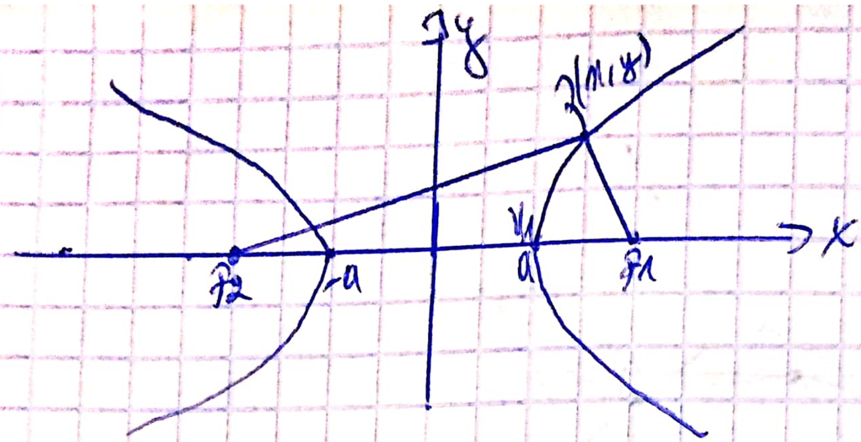


IV.3 HYPERBOLA

1. Definition: Let $P(x,y)$ be the point of the plane, we called the hyperbola of the foci F_1 and F_2 the set of point satisfying $|PF_1 - PF_2| = 2a$ with $l = 2a$, distance btn vertices and $\|\overrightarrow{F_1F_2}\| = 2c$

One axle containing foci is called focus axle.

$\boxed{c > a}$



$\|\overrightarrow{V_1V_2}\| = l = 2a$

Let $F_1(c,0)$ and $F_2(-c,0)$ and let $V_1(a,0)$ and $V_2(-a,0)$

Let $P(x,y) \in \pi$, from the definition of hyperbola

$$\left|\|\overrightarrow{PF_1}\| - \|\overrightarrow{PF_2}\|\right| = 2a$$

Let take $\|\overrightarrow{PF_1}\| - \|\overrightarrow{PF_2}\| = 2a \Rightarrow \|\overrightarrow{PF_1}\| = 2a + \|\overrightarrow{PF_2}\|$

$$\|\overrightarrow{PF_1}\| = \sqrt{(x-c)^2+(y-0)^2} = \sqrt{x^2-2c+c^2+y^2}$$

$$\|\overrightarrow{PF_2}\| = \sqrt{(x+c)^2+(y-0)^2} = \sqrt{x^2+2cx+c^2+y^2}$$

$$\|\overrightarrow{PF_1}\| = 2a + \|\overrightarrow{PF_2}\|$$

$$\sqrt{x^2-2cx+c^2+y^2} = 2a+\sqrt{x^2+2cx+c^2+y^2}$$

$$\cancel{x^2} - 2cx + \cancel{c^2} + \cancel{y^2} = 4a^2 + 4a\sqrt{x^2+2cx+c^2+y^2} + \cancel{x^2} + 2cx + \cancel{c^2} + \cancel{y^2}$$

$$-4a\sqrt{x^2+2cx+c^2+y^2} = 4a^2 + 2cx + 2cx = 4a^2 + 4cx$$

$$-a\sqrt{x^2+2cx+c^2+y^2} = \cancel{4}a^2 + cx$$

$$a^2(x^2+2cx+c^2+y^2) = a^4 + 2cxa^2 + c^2x^2$$

$$a^2x^2 + \cancel{2cxa^2} + a^2c^2 + a^2y^2 = a^4 + \cancel{2cxa^2} + c^2x^2$$

$$a^2c^2 - a^4 = c^2x^2 - a^2x^2 - a^2y^2$$

$$a^2(c^2-a^2) = (c^2-a^2)x^2 - a^2y^2$$

Let $\boxed{b^2 = c^2 - a^2}$

$$a^2b^2 = b^2x^2 - a^2y^2$$

$$1 = \frac{b^2x^2 - a^2y^2}{a^2b^2} \Rightarrow \quad a^2$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$ Hyperbola !'s cartesian equat°

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} = 1 \Rightarrow y^2 = \frac{b^2}{a^2}(x^2-a^2) \Rightarrow y = \pm\frac{b}{a}\sqrt{x^2-a^2}$$

$$\boxed{y = \frac{b}{a}\sqrt{x^2-a^2}}$$

1° domain of $x^2 - a^2 \geq 0$ $x^2 = \pm a$.

x	$-\infty$		$-a$		a		$+\infty$
x^2-a^2		$+\ +$	0	$-\ -$	0	$+\ +$	

$Dom =]-\infty, -a] \cup [a, +\infty[$

2° Asymptotes.

$$\lim_{x \to a^-} \frac{b}{a}\sqrt{x^2-a^2} = 0$$

$$\lim_{x \to a^+} \frac{b}{a}\sqrt{x^2-a^2} = 0$$

$$\lim_{x \to \pm\infty} \frac{b}{a}\sqrt{x^2-a^2} = \pm\infty$$

No vertical and Horizontal Asymptotes.

$y = mx + p$ is oblique asymptote if

$$m = \lim_{x \to \pm\infty} \frac{\frac{b}{a}\sqrt{x^2-a^2}}{x} \text{ and } p = \lim_{x \to \pm\infty} \left[\frac{b}{a}\sqrt{x^2-a^2} - mx\right]$$

$$m = \lim_{x \to +\infty} \frac{\frac{b}{a}\sqrt{x^2\left(1-\frac{a^2}{x^2}\right)}}{x} \Rightarrow \lim_{x \to +\infty} \frac{\frac{b}{a}\cancel{x}\sqrt{1-\frac{a^2}{x^2}}}{\cancel{x}} = \frac{b}{a}$$

$$m = \lim_{x \to -\infty} \frac{\frac{b}{a}\sqrt{x^2(1-\frac{a^2}{x^2})}}{x} \Rightarrow \lim_{x \to -\infty} \frac{\frac{b}{a}(-x)\sqrt{1-\frac{a^2}{x^2}}}{x} = -\frac{b}{a} //$$

$$\beta = \lim_{x \to +\infty} \left[\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x\right] \Rightarrow \lim_{x \to +\infty} \left[\frac{b}{a}x\sqrt{1-\frac{a^2}{x^2}} - \frac{b}{a}x\right]$$

$$= \lim_{x \to +\infty} \left[\frac{b}{a}x - \frac{b}{a}x\right] = 0 //$$

$$\beta' = \lim_{x \to -\infty} \left[\frac{b}{a}\sqrt{x^2-a^2} + \frac{b}{a}x\right] = \lim_{x \to -\infty} \left[\frac{b}{a}(-x)\sqrt{1-\frac{a^2}{x^2}} + \frac{b}{a}x\right]$$

$$= \lim_{x \to -\infty} \left[-\frac{b}{a}x + \frac{b}{a}x\right] = 0 //$$

$y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are oblique Asymptots.

3- Derivative and variation

$$y' = \left(\frac{b}{a}\sqrt{x^2-a^2}\right)' = \frac{2\frac{b}{a}x}{2\sqrt{x^2-a^2}} = \frac{\frac{b}{a}x}{\sqrt{x^2-a^2}}$$

	$-\infty$	$-a$	0	a	$+\infty$
$\frac{b}{a}x$	−	−	0	+	+
$\sqrt{x^2-a^2}$	+ +	0	+	+ 0	+ +
y'	−	||	0	+ 0	+ + +
y	$+\infty \searrow$	0		|| 0	$\nearrow +\infty$

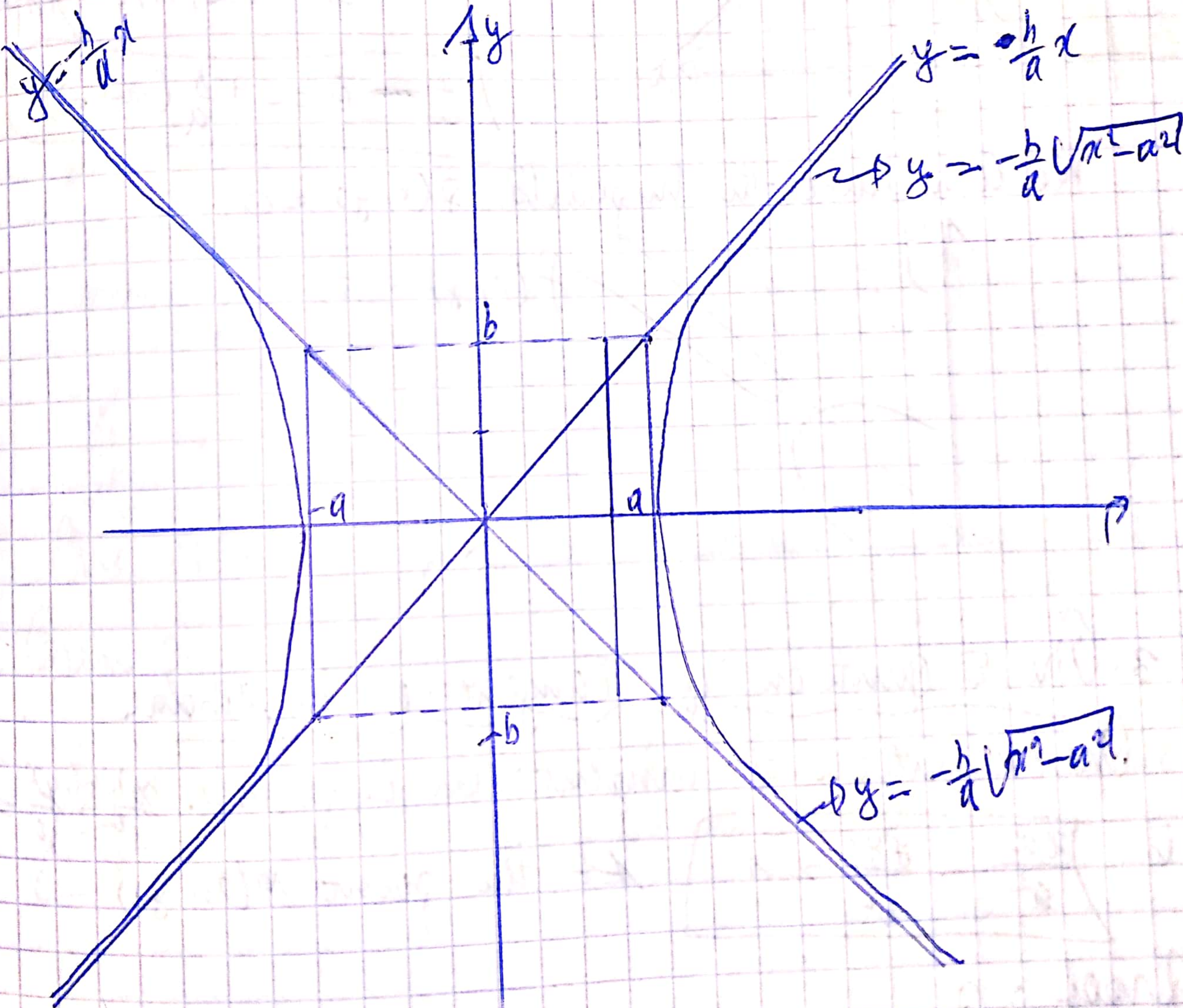
$$y'(a) = \frac{\frac{b}{a} a}{\sqrt{a^2 - a^2}} = \frac{b}{0} = \pm \infty$$

$$y'(-a) = \frac{\frac{b}{a}(-a)}{\sqrt{a^2 - a^2}} = \frac{-b}{0} = \pm \infty$$

$x = a$ and $x = -a$ are vertical tangent

$y = -\frac{b}{a}x$ if $x = 0$, $y = 0$, if $x = a$, $y = -b$

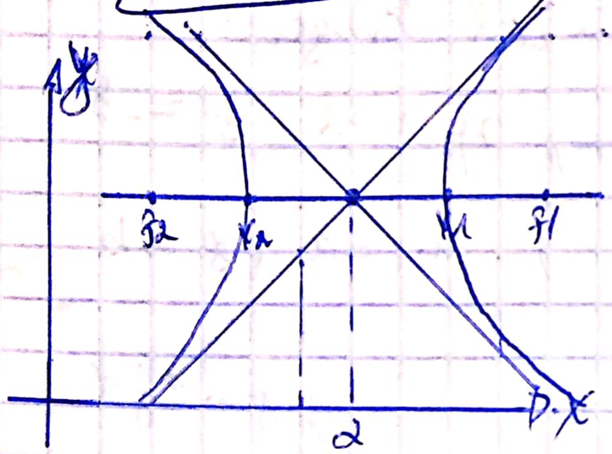
$y = \frac{b}{a}x$ if $x = 0$, $y = 0$, if $x = a$, $y = b$



2. Shifted Hyperbola

The Equation of hyperbola centred on $C(\alpha, \beta)$ is

$$\boxed{\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1}$$



$F_1 = (c+\alpha, \beta)$

$F_2 = (-c+\alpha, \beta)$

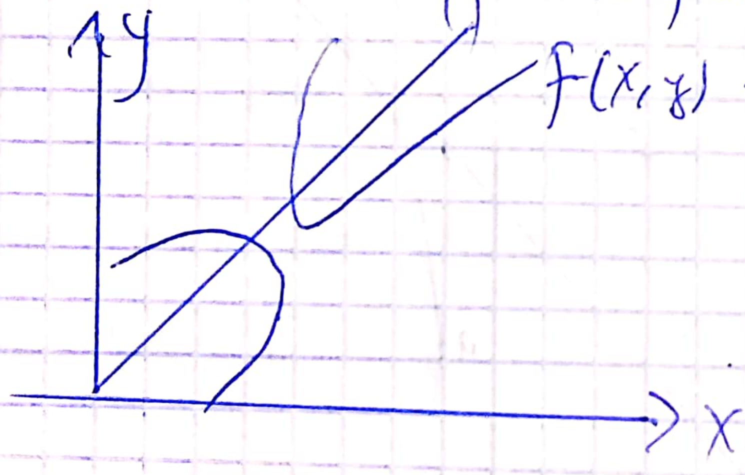
$V_1 = (a+\alpha, \beta)$

$V_2 = (-a+\alpha, \beta)$

Asymptotes.

$$\boxed{y - \beta = \pm \frac{b}{a}(x-\alpha)}$$

We can have the hyperbola $f(x,y)=0$



3. The Equation of Tangent on Hyperbola.

The equation of tangent on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\boxed{\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1}$ At the point $M(x_1, y_1)$

Proof: The equation of tangent at $M(x_1, y_1)$ is

$$y - y_1 = y'_M (x - x_1)$$

where y'_M is the slope of the tangent on hyperbola.

~~Jet~~ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola.

$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)' = (1)' \Rightarrow \frac{2x}{a^2} - \frac{2y'y}{b^2} = 0 \Rightarrow y' = \frac{2b^2x}{2a^2y}$

$y' = \frac{b^2x}{a^2y}$ and $y'_M = \frac{b^2x_1}{a^2y_1}$

The slope $y'_M = \frac{b^2x_1}{a^2y_1}$ is the tangent gives.

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$a^2y_1(y - y_1) = b^2x_1(x - x_1)$$

$$a^2y_1y - a^2y_1 = b^2x_1x - b^2x_1^2$$

$$b^2x_1^2 - a^2y_1^2 = b^2x_1x - a^2y_1y$$

$$a^2b^2\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) = a^2b^2\left(\frac{x_1x}{a^2} - \frac{y_1y}{b^2}\right)$$

$$\frac{x_1^2}{a^2} = \frac{x_1x}{a^2} - \frac{y_1y}{b^2}$$

The point $M(x_1, x_2)$ belongs to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Then $\frac{x_1^2}{a^2} - \frac{y^2}{b^2} = 1$ Therefore $1 = \frac{x_1x}{a^2} - \frac{y_1y}{b^2}$

$\boxed{\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1}$ is the equation of tangent on hyperbola.

Scanned with CamScanner

4- parametric equation on hyperbola

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the cartesian equation of hyperbola,

The parametric equation of hyperbola are

1° $\begin{cases} x=\frac{a}{\cos t} \\ y=b\tan t \end{cases}$ 2° $x=a\,cht$ (hyperbolic cosinus (cht))

$y=b\,sht$ (hyperbolic sine (sht).

3° $\begin{cases} x=\frac{a}{2}\left(t+\frac{1}{t}\right) \\ y=\frac{b}{2}\left(t-\frac{1}{t}\right) \end{cases}$

Proof: 1° $\begin{cases} x=\frac{a}{\cos t} \\ y=b\tan t \end{cases}$ $\Rightarrow$ in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ gives.

$$\frac{\frac{a^2}{\cos^2 t}}{a^2}-\frac{b^2\tan^2 t}{b^2} \Rightarrow \frac{1}{\cos^2 t}-\frac{1-\sin^2 t}{\cos^2 t}=\frac{\cos^2 t}{\cos^2 t}=1$$

2° $\begin{cases} x=a\,cht \\ y=b\,sht \end{cases}$ in the equation of hyperbola.

$$\frac{a^2 ch^2 t}{a^2}-\frac{a^2 sh^2 t}{b^2}=ch^2 t-sh^2 t=1$$

3^o $\begin{cases} x = \frac{a}{2}\left(t + \frac{1}{t}\right) \\ y = \frac{b}{2}\left(t - \frac{1}{t}\right) \end{cases}$ in the hyperbolic

$$\frac{\left[\frac{a}{2}\left(t+\frac{1}{t}\right)\right]^2}{a^2} - \frac{\left[\frac{b}{2}\left(t-\frac{1}{t}\right)\right]^2}{b^2}$$

$$\frac{\frac{\cancel{a^2}}{4}\left(t^2+2+\frac{1}{t^2}\right)}{\cancel{a^2}} - \frac{\frac{\cancel{b^2}}{4}\left(t^2-2+\frac{1}{t^2}\right)}{\cancel{b^2}}$$

$$\frac{t^2}{4} + \frac{2}{4} + \frac{1}{4t^2} - \frac{t^2}{4} - \frac{2}{4} + \frac{1}{4t^2} = \frac{4}{4} = 1$$

5. Application

① Let $9x^2 - 16y^2 - 36x - 32y - 124 = 0$ and $2x^2 + 3y^2 - 8x + 6y - 7 = 0$ be two equation of conics

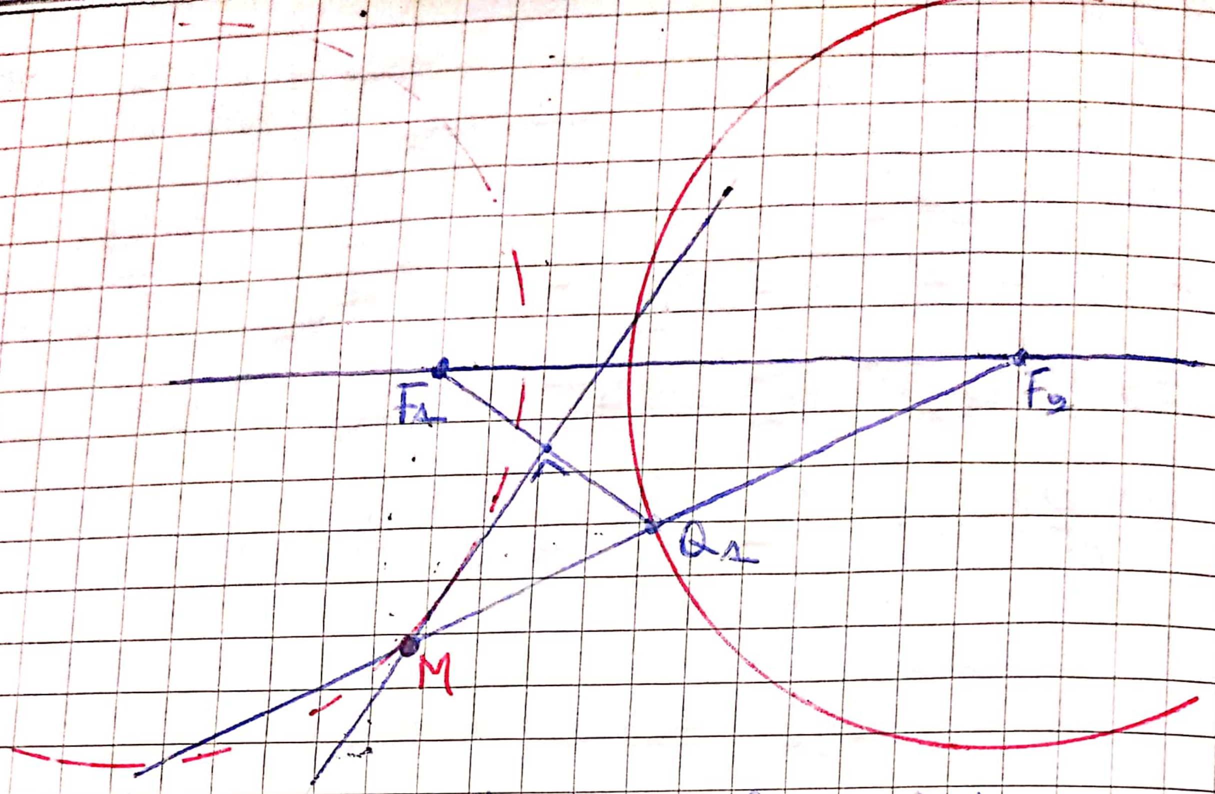
ⓐ Reduce to canonical form

ⓑ Find the vertices, foci and asymptotes if possible.

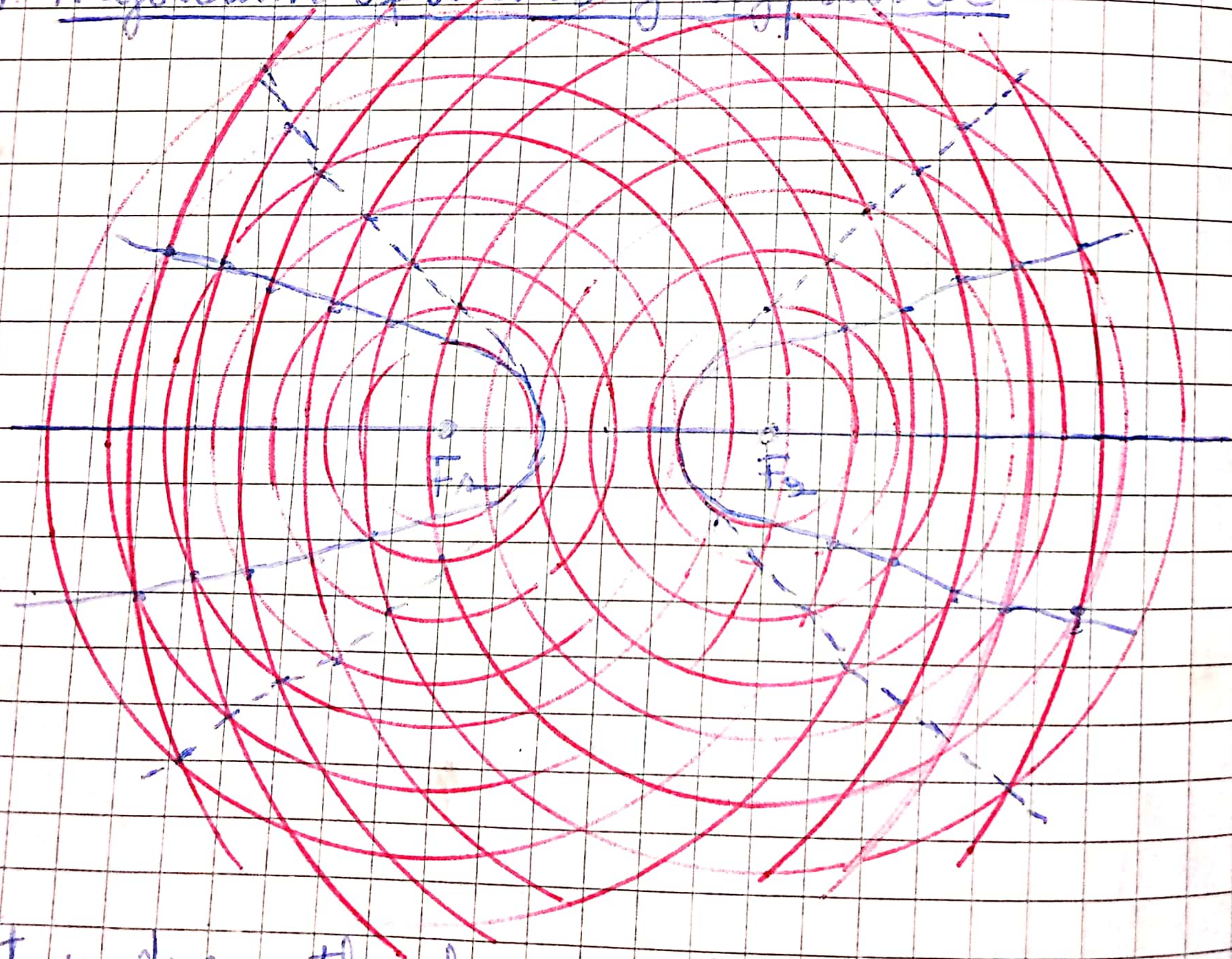
~~On Same que~~

ⓒ Draw the conics.

② Find the equation of conics centred on $C(4,-1)$ of the) focus $F(1,-1)$ and passing through the point $P(8,0)$



D Algorithm of drawing hyperbola



1 Let us draw the focus axle where F_1 and F_2 are
2. Let us draw the concentric circle of different radius around F_1 and F_2
3 the intersection of that concentric circle out of focus distance will give ideal of drawing

the hyperbola

4. We joint point by point to get the needed hyperbola

5. We have more than one hyperbola

E. Graphic study of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Cfr 5e)

$Dom\, f =]-\infty, -a] \cup [a, +\infty[$

N.B ① the point $(a,0)$ and $(-a,0)$ are the vertices of the hyperbola

② the focus we have the coordinates $F_1(c,0)$ and $F_2(-c,0)$

F. Equation of the tangent of the hyperbola

Let consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (*)

Let $P(x_1, y_1)$ be the point of the hyperbola

We are asked to find the tangent T on the hyperbola.

After differentiating both side of (*) we have

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow 2yy' = \frac{2b^2x}{a^2}$$

$$\Rightarrow yy' = \frac{b^2}{a^2}x \Rightarrow y' = \frac{b^2x}{a^2y} \Rightarrow y'_M = \frac{b^2x_1}{a^2y_1}$$

$$\boxed{T \equiv y - y_1 = y'_M(x - x_1)}$$

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1) \Rightarrow$$ [illegible]

3^o $\begin{cases} x = \frac{a}{2}\left(t + \frac{1}{t}\right) \\ y = \frac{b}{2}\left(t - \frac{1}{t}\right) \end{cases}$ in the hyperbolic

$$\frac{\left[\frac{a}{2}\left(t+\frac{1}{t}\right)\right]^2}{a^2} - \frac{\left[\frac{b}{2}\left(t-\frac{1}{t}\right)\right]^2}{b^2}$$

$$\frac{\frac{\cancel{a^2}}{4}\left(t^2+2+\frac{1}{t^2}\right)}{\cancel{a^2}} - \frac{\frac{\cancel{b^2}}{4}\left(t^2-2+\frac{1}{t^2}\right)}{\cancel{b^2}}$$

$$\frac{t^2}{4} + \frac{2}{4} + \frac{1}{4t^2} - \frac{t^2}{4} - \frac{2}{4} + \frac{1}{4t^2} = \frac{4}{4} = 1$$

5. Application

① Let $9x^2 - 16y^2 - 36x - 32y - 124 = 0$ and $2x^2 + 3y^2 - 8x + 6y - 7 = 0$ be two equation of conics

ⓐ Reduce to canonical form

ⓑ Find the vertices, foci and asymptotes if possible.

~~One same ques~~

ⓒ Draw the conics.

② Find the equation of conics centred on $C(4,-1)$ of the focus $F(1,-1)$ and passing through the point $P(8,0)$

Scanned with CamScanner

③ find the center, foci, vertices and draw the conic find the parametric equation

$$9x^2 + 16y^2 - 36x + 96y + 36 = 0$$

④ find the equation of the ellipse E which is centred on C(1,1) of the focus C(6,2) and passing through A(4,6)

⑤ find the equation of the tangent on the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 0$$ i) parallel to $x + 2y - 1 = 0$

ii) orthogonal to $3x - 5y + 10 = 0$

Solution

①

$$9x^2 - 16y^2 - 36x - 32y - 124 = 0$$

$$9(x^2 - 4x) - 16(y^2 - 2y) - 124 = 0$$

$$9[(x-2)^2 - 4] - 16[(y-1)^2 - 1] - 124 = 0$$

$$9(x-2)^2 - 36 - 16(y-1)^2 + 16 - 124 = 0$$

$$9(x-2)^2 - 16(y-1)^2 - 144 = 0$$

$$\frac{(x-2)^2}{\frac{144}{9}} - \frac{(y-1)^2}{\frac{144}{16}} = 1$$

$$\frac{(x-2)^2}{\left(\frac{12}{3}\right)^2} - \frac{(y-1)^2}{\left(\frac{12}{4}\right)^2} = 1 \Rightarrow \frac{(x-2)^2}{4^2} - \frac{(y-1)^2}{3^2} = 1$$

Hyperbola centred on C(2,1), $a = 4$, $b = 3$

$$b^2 = c^2 - a^2 \Rightarrow c^2 = b^2 + a^2$$

$$= 16 + 9$$

$$= 25$$

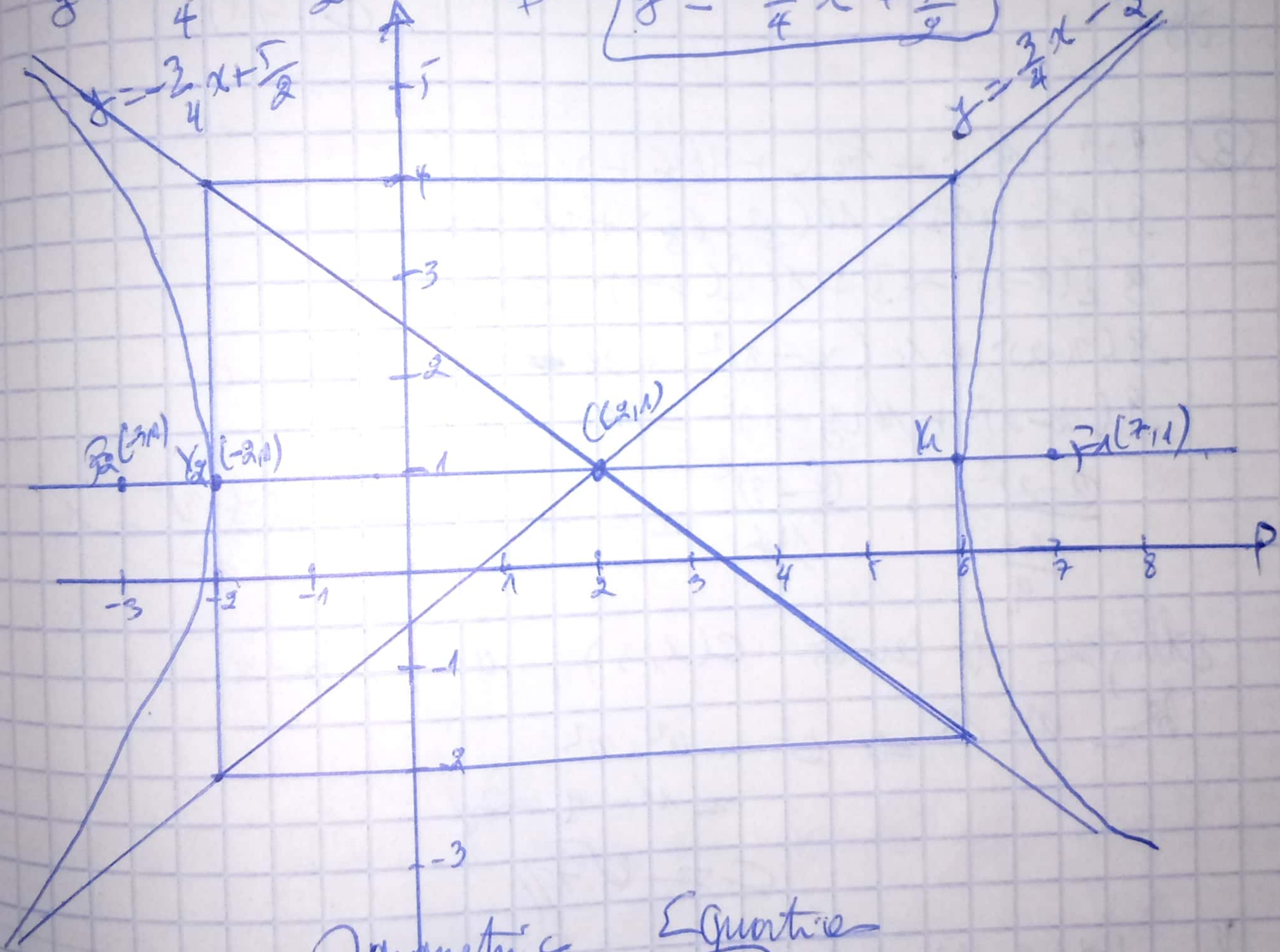
$V_1 = (\alpha + a, \beta) = (2+4, 1) = (6, 1)$
$V_2 = (x - a, \beta) = (2-4, 1) = (-2, 1)$
$F_1 = (\alpha + c, \beta) = (2+5, 1) = (7, 1)$
$F_2 = (\alpha - c, \beta) = (2-5, 1) = (-3, 1)$

$$y - \beta = \pm \frac{b}{a}(x - \alpha)$$

$$y - 1 = \pm \frac{3}{4}(x - 2)$$

$$y = \frac{3}{4}x - \frac{3}{2} + 1 \Rightarrow \boxed{y = \frac{3}{4}x - \frac{1}{2}}$$

$$y = -\frac{3}{4}x + \frac{3}{2} + 1 \Rightarrow \boxed{y = -\frac{3}{4}x + \frac{5}{2}}$$



Parametric Equation

$$1^o - \begin{cases} x = \frac{a}{2}\left(t + \frac{1}{t}\right) \\ y = \frac{b}{2}\left(t - \frac{1}{t}\right) \end{cases} \Rightarrow \begin{cases} x = 2\left(t + \frac{1}{t}\right) \\ y = \frac{3}{2}\left(t - \frac{1}{t}\right) \end{cases}$$

t is parameter

2° $\begin{cases} x = \frac{a}{\cos t} \\ y = b\tan t \end{cases} \Rightarrow \begin{cases} x = \frac{4}{\cos t} \\ y = 3\tan t \end{cases}$ || $\begin{cases} x = a\,\mathrm{ch}\,t \\ y = b\,\mathrm{sh}\,t \end{cases}$

N.B The centre of conics can be obtained by solving the system $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$

$\frac{\partial f}{\partial x} = 0 \Rightarrow \begin{cases} 18x - 36 = 0 \\ -32y + 32 = 0 \end{cases} \Rightarrow x = 2, \; y = 1$
$\frac{\partial f}{\partial y} = 0 \Rightarrow$

$C(2,1)$

③ $9x^2 + 16y^2 - 36x + 96y + 36 = 0$

$9(x^2 - 4x) + 16(y^2 - 6y) + 36 = 0$

$9\{(x-2)^2 - 4\} + 16\{(y-3)^2 - 9\} + 36 = 0$

$9(x-2)^2 + 16(y-3)^2 + 36 - 144 + 36 = 0$

$9(x-2)^2 + 16(y-3)^2 = 144$

$\frac{(x-2)^2}{\frac{144}{9}} + \frac{(y-3)^2}{\frac{144}{16}} = 1 \Rightarrow \frac{(x-2)^2}{4^2} + \frac{(y-3)^2}{3^2} = 1$

Ellipse of center $C(2,3)$, $a = 4$, $b = 3$.

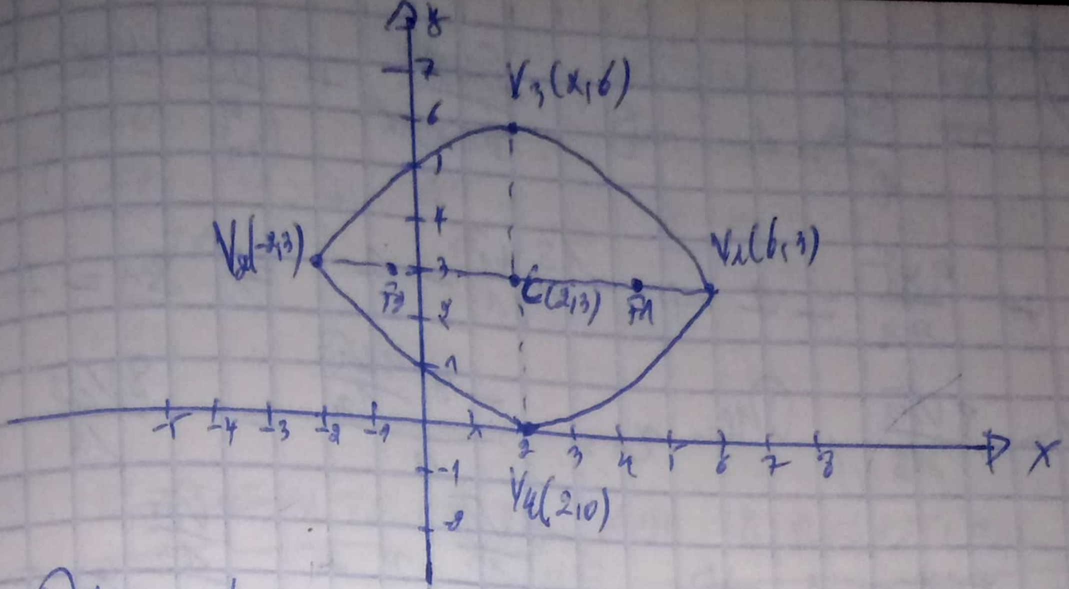
$b^2 = a^2 - c^2 \Rightarrow c^2 = a^2 - b^2$
$= 16 - 9 = 7$

$c = \sqrt{7}$

$V_1 = (\alpha + a, \beta) = (2+4, 3) = (6,3)$
$V_2 = (\alpha - a, \beta) = (2-4, 3) = (-2,3)$
$V_3(\alpha, \beta + b) = (2, 3+3) = (2,6)$
$V_4(\alpha, \beta - b) = (2, 3-3) = (2,0)$

$F_1 = (\alpha + c, \beta) = (2+\sqrt{7}, 3)$

$F_2 = (\alpha - c, \beta) = (2-\sqrt{7}, 3)$



Parametric equation

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \Rightarrow \begin{cases} x = 4\cos t \\ y = 3\sin t \end{cases}$$

(5) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

The equation of tangent at $P(x_1, y_1)$ is:

$$\frac{xx_1}{16} + \frac{yy_1}{9} = 1 \Rightarrow y = -\frac{9x_1}{16y_1}x + \frac{9}{y_1}$$

(a) Tangent parallel to $x + 2y - 1 = 0$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

$$m' = -\frac{1}{2}$$

Two straight lines are parallel if

$$m = m' \Rightarrow \frac{9x_1}{16y_1} = -\frac{1}{2}$$

$$x_1 = -\frac{8y_1}{9}$$

$P(x_1, y_1)$ is the point of ellipse.

$$\frac{x_1^2}{16} + \frac{y_1^2}{9} = 1$$

$$\frac{\left(-\frac{8y_1}{9}\right)^2}{16} + \frac{y_1^2}{9} = 1 \Rightarrow \frac{64y_1^2}{81\times 16} + \frac{y_1^2}{9} = 1$$

$$\frac{4y_1^2}{81} + \frac{9y_1^2}{81} = 1 \Rightarrow \frac{13y_1^2}{81} = 1 \Rightarrow y_1 = \pm\frac{9\sqrt{13}}{13}$$

If $y_1 = \frac{9\sqrt{13}}{13}$, then $x_1 = \frac{-8\frac{9\sqrt{13}}{13}}{9} = -\frac{8\sqrt{13}}{13}$

$$\frac{x_1x}{16} + \frac{y_1y}{9} = 1 \Rightarrow \frac{-\frac{8\sqrt{13}}{13}}{16}x + \frac{\frac{9\sqrt{13}}{13}}{9}y = 1$$

$$\boxed{-\frac{\sqrt{13}}{26}x + \frac{\sqrt{13}}{13}y - 1 = 0}$$

If $y_1 = \frac{-9\sqrt{13}}{13}$, then $x_1 = \frac{-8\frac{-9\sqrt{13}}{13}}{9} = \frac{8\sqrt{13}}{13}$

$$\frac{x_1x}{16} + \frac{y_1y}{9} = 1 \Rightarrow \frac{\frac{8\sqrt{13}}{13}}{16}x - \frac{\frac{9\sqrt{13}}{13}}{9}y = 1$$

$$\boxed{\frac{\sqrt{13}}{26}x - \frac{\sqrt{13}}{13}y - 1 = 0}$$

(b) Tangent is orthogonal to $3x-5y+10=0$

$5y = 3x+10 \Rightarrow y = \frac{3}{5}x + 2$ $\quad m' = \frac{3}{5}$

Two straight lines are orthogonal if

$mm' = -1 \Rightarrow \frac{9x_1}{16y_1}\cdot\frac{3}{5} = -1 \Rightarrow 27x_1 = -80y_1$

$x_1 = -\frac{80}{27}y_1$

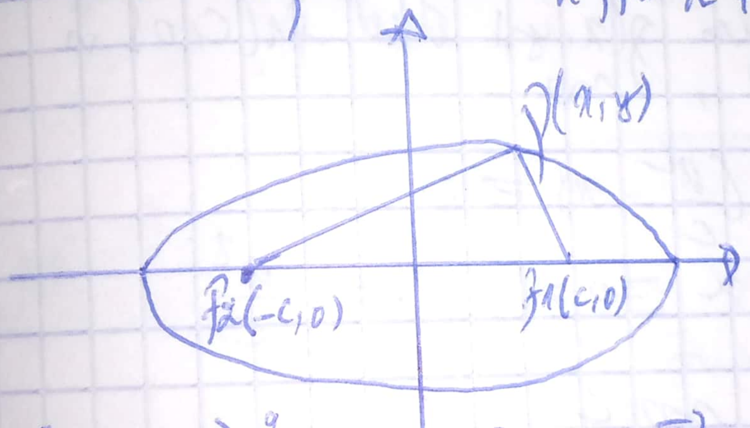
$p(x_1, y_1)$ is the point of ellipse $\frac{x_1^2}{16} + \frac{y_1^2}{9} = 1$

$$\frac{\left(\frac{80}{27} y_1\right)}{16} + \frac{y_1}{9} = 1$$

IV ECCENTRICITY AND DIRECTION STRAIGHT LINE

1. Case of Ellipse

Let consider the ellipse at the foci $F_1(c, 0)$ and $F_2(-c, 0)$ let $p(x, y)$ be the point of ellipse from the definition $\|\vec{pF_1}\| + \|\vec{pF_2}\| = 2a$



$$\|\vec{pF_1}\|^2 - \|\vec{pF_2}\|^2 = (\|\vec{pF_1}\| + \|\vec{pF_2}\|)(\|\vec{pF_1}\| - \|\vec{pF_2}\|)$$

$$\|\vec{pF_1}\|^2 = (x-c)^2 + (y-0)^2 = x^2 - 2cx + c^2 + y^2$$

$$\|\vec{pF_2}\| = (x+c)^2 + (y-0)^2 = x^2 + 2cx + c^2 + y^2$$

$$\|\vec{pF_1}\|^2 - \|\vec{pF_2}\|^2 = (x^2 - 2cx + c^2 + y^2) - (x^2 + 2cx + c^2 + y$$

$$= x^2 - 2cx + c^2 + y^2 - x^2 - 2cx - c^2 - y^2$$

$$= -4cx.$$

$$\boxed{\|\vec{pF_1}\| - \|\vec{pF_2}\|^2 = -4cx}$$

From $\|\vec{PF_1}\|^2 - \|\vec{PF_2}\|^2 = (\|\vec{PF_1}\| + \|\vec{PF_2}\|)(\|\vec{PF_1}\| - \|\vec{PF_2}\|)$
with $\|\vec{PF_1}\| + \|\vec{PF_2}\| = 2a$ is

$$2a(\|\vec{PF_1}\| - \|\vec{PF_2}\|) = -4cx$$

$$\|\vec{PF_1}\| - \|\vec{PF_2}\| = -\frac{4cx}{2a} = -\frac{2cx}{a}$$

We can solve the system

$$\|\vec{PF_1}\| - \|\vec{PF_2}\| = -\frac{2cx}{a}$$
$$\|\vec{PF_1}\| + \|\vec{PF_2}\| = 2a$$

$$2\|\vec{PF_1}\| = 2a - \frac{2cx}{a}$$

$$\|\vec{PF_1}\| = a - \frac{cx}{a} = \frac{c}{a}\left(\frac{a^2}{c} - x\right)$$

The distance between $P(x,y)$ and $F_1(c,0)$ is greater than 0, so

$$\|\vec{PF_1}\| \geq 0 \Rightarrow \frac{c}{a}\left(\frac{a^2}{c} - x\right) \geq 0$$

$$\frac{c}{a} > 0 \text{ and } \frac{a^2}{c} - x \geq 0 \Rightarrow x = \frac{a^2}{c}$$

For ellipse $a > c$ so $\frac{c}{a} < 1$.

By the definition: the eccentricity $e = \frac{c}{a}$
and the direction line which is corresponding to the focus F_1 is $x = \frac{a^2}{c}$

We can solve

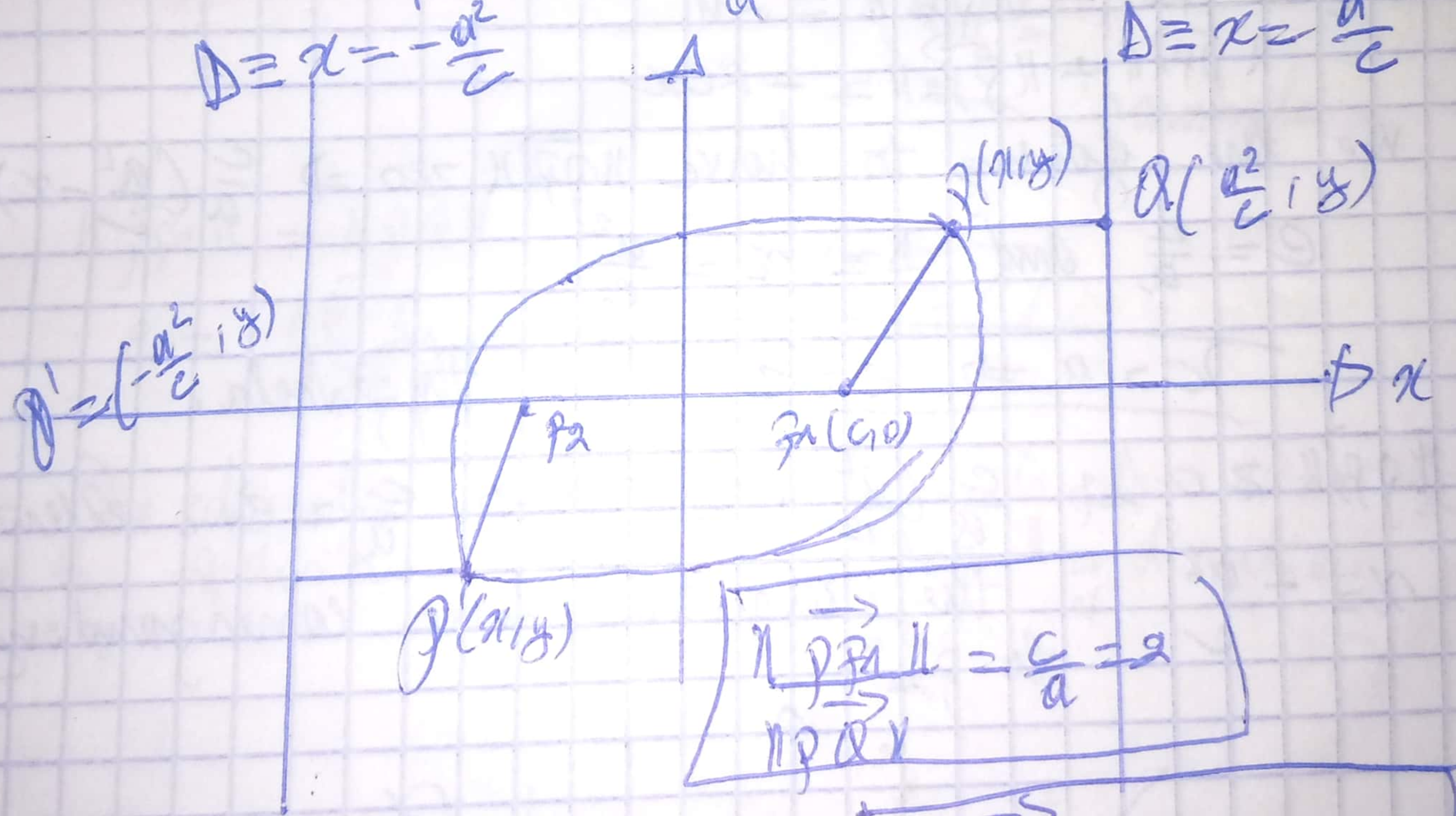
$$-\|\vec{PF_1}\| + \|\vec{PF_2}\| = 2cx$$
$$\|\vec{PF_1}\| + \|\vec{PF_2}\| = 2a$$

$$2\|\vec{PF_2}\| = 2a + \frac{2cx}{a}$$

$$\|\overrightarrow{PF_2}\| = \frac{c}{a}\left(\frac{a^2}{c} + x\right) = \frac{c}{a} = 0$$
$$x = -\frac{a^2}{c}$$

The eccentricity is $e = \frac{c}{a}$ and the direction line corresponding to F_2 is $x = -\frac{a^2}{c}$

Defi: For ellipse of the focus corresponding to the direction straight line $\Delta \equiv x = \frac{a^2}{c}$ the ratio between $\|\overrightarrow{PF_1}\|$ and $\|\overrightarrow{PQ}\|$ with Q the perpendicular point from P to the direction straight is constant and equal eccentricity $e = \frac{c}{a}$



$$\boxed{\frac{\|\overrightarrow{PF_1}\|}{\|\overrightarrow{PQ}\|} = \frac{c}{a} = a}$$

or $$\boxed{\frac{\|\overrightarrow{PF_2}\|}{\|\overrightarrow{PQ}\|} = e = \frac{c}{a}}$$

$PF_2 = a + \frac{xc}{a}$ and for $PF_1 = a - \frac{xc}{a}$

Hence $\begin{cases} PF_1 = a - \frac{xc}{a} \\ PF_2 = a + \frac{xc}{a} \end{cases}$

$\Rightarrow PF_1 = a - \frac{xc}{a} = \frac{c}{a}\left(\frac{a^2}{c} - x\right) > 0$

PF_1 is the distance from P of ellipse to the focus is strictly positive $PF_1 > 0$

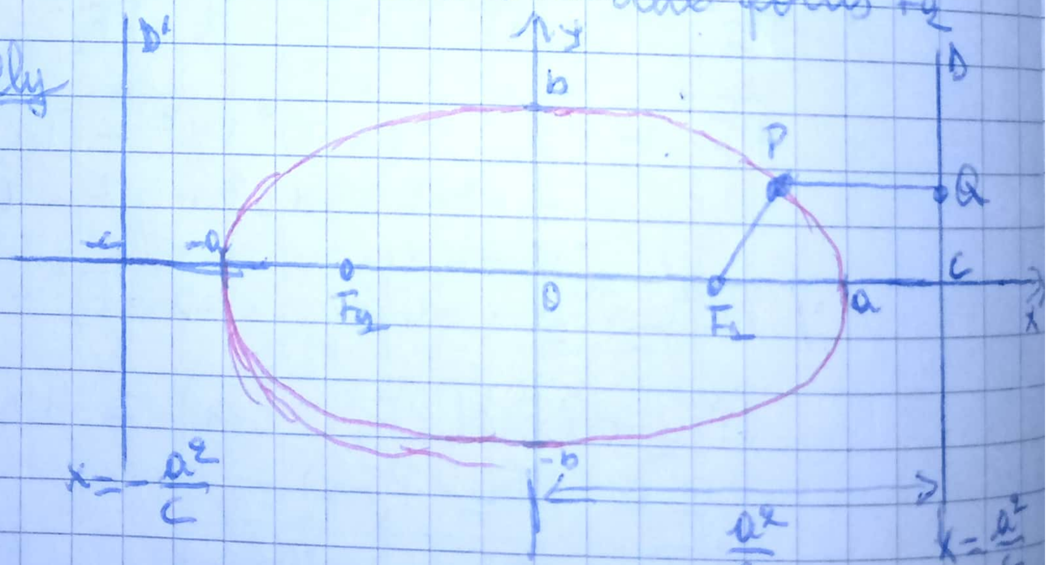
$\Rightarrow \frac{c}{a}\left(\frac{a^2}{c} - x\right) > 0 \Rightarrow \frac{a^2}{c} - x > 0$ because $\frac{c}{a}$ is positive

• If we consider the straight line of the equation $x = \frac{a^2}{c}$ we saw that $x = \frac{a^2}{c}$ is on the right of all points of the ellipse E

• conclusion 1. $\boxed{D \equiv x = \frac{a^2}{c}}$ is called the director associated with the focus F_1

2. $\boxed{D' \equiv x = -\frac{a^2}{c}}$ is called the director associated with the focus F_2

Graphically



Definitions: The ratio $\frac{PF_1}{PQ}$ is a constant equal to $\frac{c}{a}$

$\Rightarrow \boxed{\frac{PF_1}{PQ} = \frac{c}{a}}$ is called the excentricity of the ellipse

$\Rightarrow$ we write the excentricity $\boxed{e = \frac{c}{a}}$ as from the ellips $a > c$ then $\boxed{0 < e < 1}$

Definition 2: the ellipse is the set of points where the ratio of the distance on the focus and the distance corresponding directrix is the constant equal to the excentricity.

b. Case of the hyperbola
Let consider the hyperbola of equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a < c$ for all point P of

$|\overline{PF_1} - \overline{PF_2}| = 2a$

$\overline{PF_1}^2 = (x-c)^2 + y^2$ and $\overline{PF_2}^2 = (x+c)^2 + y^2$

and $\overline{PF_1}^2 - \overline{PF_2}^2 = (\overline{PF_1} - \overline{PF_2})(\overline{PF_1} + \overline{PF_2}) = -4cx$

$\overline{PF_1} + \overline{PF_2} = \frac{-4cx}{\overline{PF_1} - \overline{PF_2}} = \frac{-4cx}{2a} = -\frac{2cx}{a}$

$\overline{PF_2} + \overline{PF_1} = \frac{2cx}{a}$

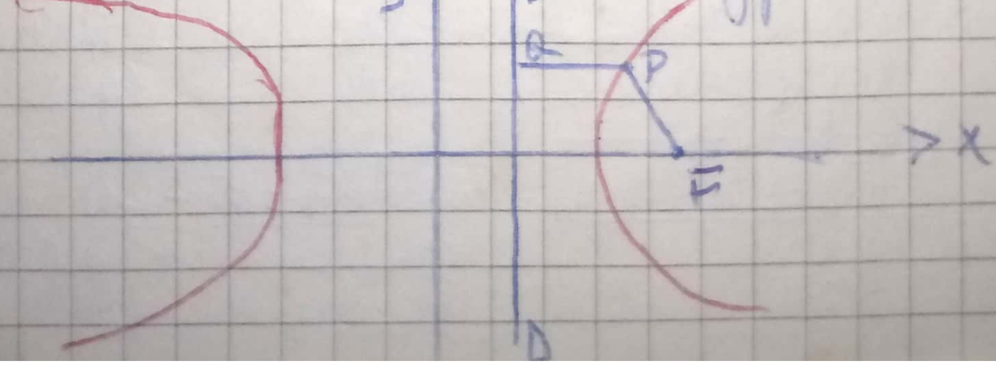
$PF_1 - \overline{PF_2} = 2a$

$PF_2 = \frac{xc}{a} + a$ and $PF_1 = \frac{xc}{a} - a$

$PF_2 = \frac{c}{a}\left(x + \frac{a^2}{c}\right)$ and $PF_1 = \frac{c}{a}\left(x - \frac{a^2}{c}\right)$

using the same manner as did on ellips

$\boxed{\frac{PF}{PQ} = \frac{c}{a} = e}$ is the excentricity on the hyperbola



2. Case of Hyperbola.

Let $P(x, y)$ be only point of hyperbola of the focus $F_1(c, 0)$ and $F_2(-c, 0)$ and, let

$$\|\vec{PF_1}\| - \|\vec{PF_2}\| = 2a$$

be the geometric equation of hyperbola.

from $\|\vec{PF_1}\| - \|\vec{PF_2}\| = 2a$

$$\|\vec{PF_1}\|^2 - \|\vec{PF_2}\|^2 = (\|\vec{PF_1}\| + \|\vec{PF_2}\|)(\|\vec{PF_1}\| - \|\vec{PF_2}\|)$$

we have $\|\vec{PF_1}\| - \|\vec{PF_2}\| = 2a$

$$2a(\|\vec{PF_1}\| + \|\vec{PF_2}\|) = -4cx$$

$$\Rightarrow \|\vec{PF_1}\| - \|\vec{PF_2}\| = 2a$$
$$\|\vec{PF_1}\| + \|\vec{PF_2}\| = -2cx$$

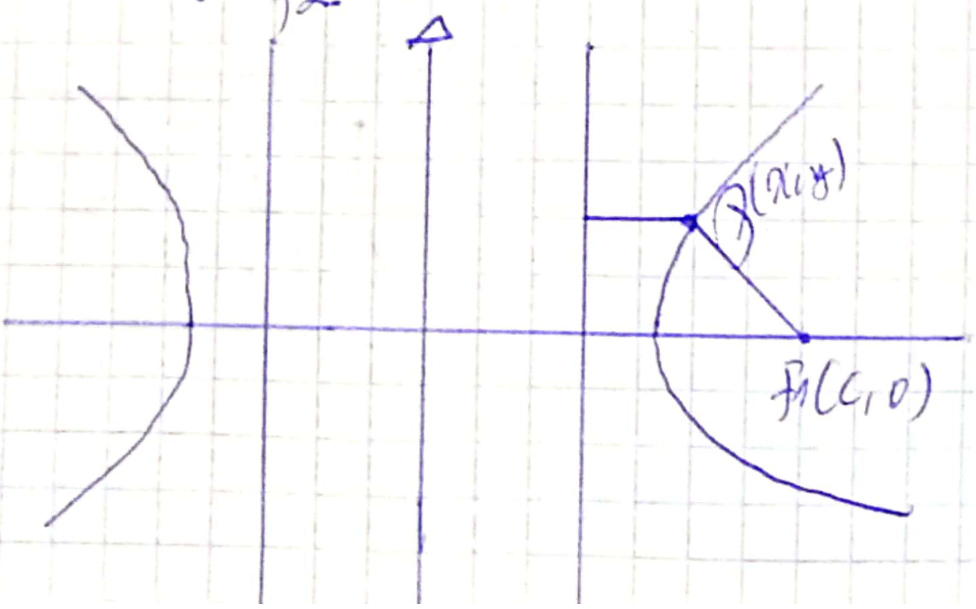
we are going to have $\|\vec{PF_1}\| \geq 0 \Rightarrow \frac{c}{a}\left(\frac{a^2}{c} - x\right) \geq 0$

$e = \frac{c}{a}$ and $D \equiv x = \frac{a^2}{c}$

$\boxed{c > a \Rightarrow e > 1}$ for Hyperbola.

$\|\vec{PF_2}\| \geq 0 \Rightarrow \frac{c}{a}\left(\frac{a^2}{c} + x\right) \geq 0 \Rightarrow \frac{c}{a} = e$ is eccentricity

$x = -\frac{a^2}{c}$ is the direction line corresponding to F_2.



2. Case of Hyperbola.

Let $P(x,y)$ be any point of hyperbola of the focus $F_1(c,0)$ and $F_2(-c,0)$ and, let

$$\|\overrightarrow{PF_1}\| - \|\overrightarrow{PF_2}\| = 2a$$

be the geometric equation of hyperbola.

From $\begin{cases} \|\overrightarrow{PF_1}\| - \|\overrightarrow{PF_2}\| = 2a \\ \|\overrightarrow{PF_1}\|^2 - \|\overrightarrow{PF_2}\|^2 = (\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\|)(\|\overrightarrow{PF_1}\| - \|\overrightarrow{PF_2}\|) \end{cases}$

We have $\|\overrightarrow{PF_1}\| - \|\overrightarrow{PF_2}\| = 2a$

$$2a(\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\|) = -4cx$$

$$\Rightarrow \begin{cases} \|\overrightarrow{PF_1}\| - \|\overrightarrow{PF_2}\| = 2a \\ \|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\| = -2cx \end{cases}$$

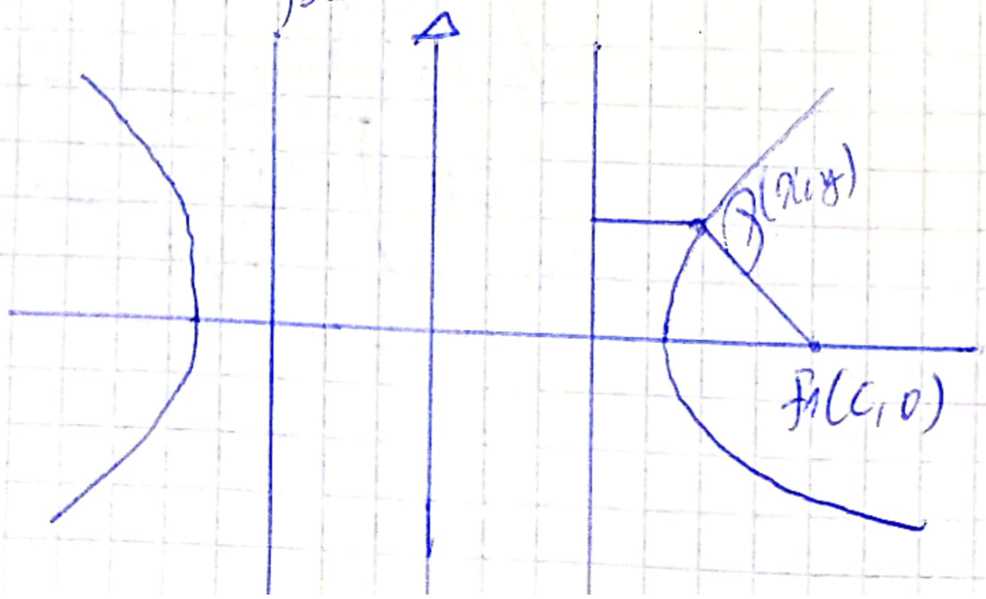
We are going to have $\|\overrightarrow{PF_1}\| \geq 0 \Rightarrow \frac{c}{a}\left(\frac{a^2}{c} - x\right) \geq 0$

$e = \frac{c}{a}$ and $D \equiv x = \frac{a^2}{c}$

$\boxed{c > a \Rightarrow e > 1}$ for hyperbola.

$\|\overrightarrow{PF_2}\| \geq 0 \Rightarrow \frac{c}{a}\left(\frac{a^2}{c} + x\right) \geq 0 \Rightarrow \frac{c}{a} = e$: eccentricity

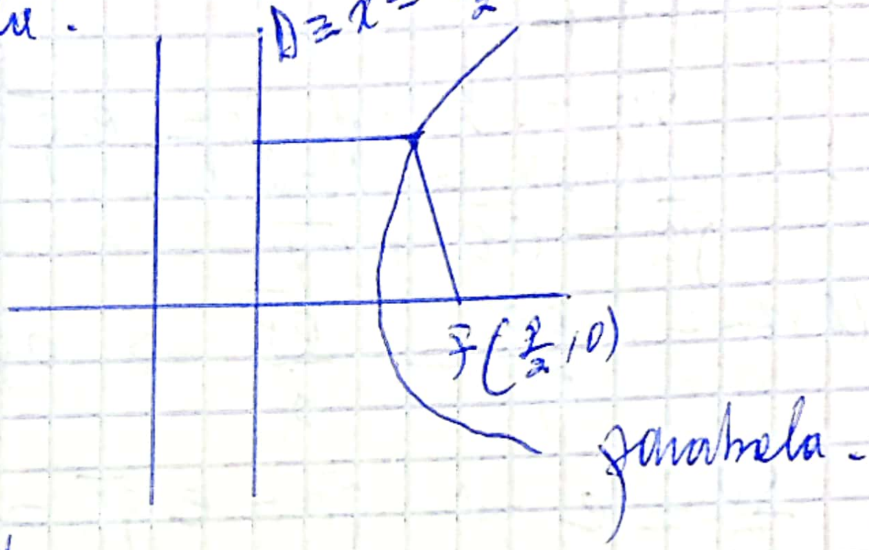
$x = -\frac{a^2}{c}$ is the direction line corresponding to F_2.



$$\frac{\|\vec{PF}\|}{\|\vec{PQ}\|} = \frac{c}{a} = \text{eccentricity}.$$

3. <u>Case of parabola</u>

Let $P(x,y)$ be any point of parabola and $D \equiv x = \frac{P}{2}$ be the direction straight line on parabola.



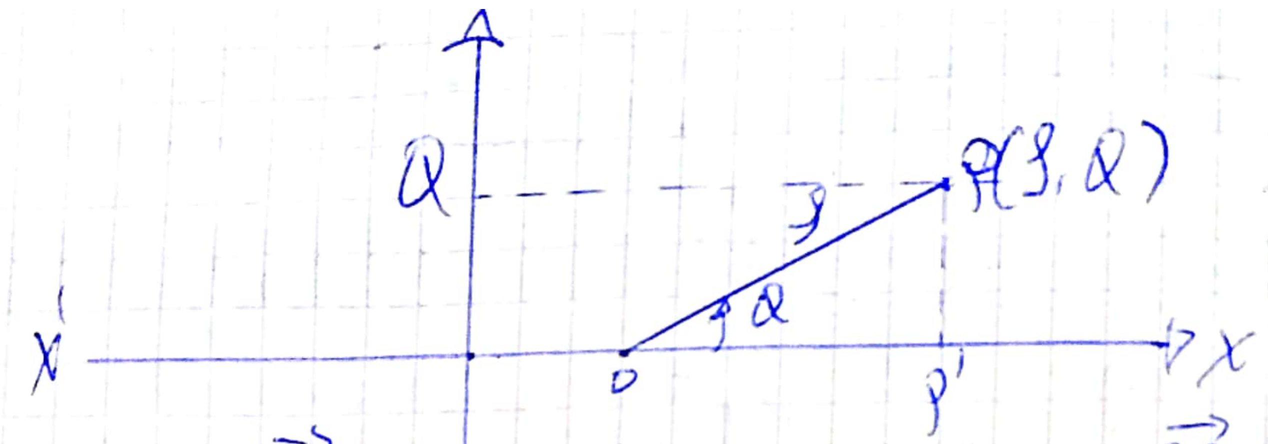
$\|\vec{PF}\| = \|\vec{PQ}\|$

$$e = \frac{\|\vec{PF}\|}{\|\vec{PQ}\|} = 1 \qquad \boxed{e = 1}$$

The direction straight line is $D = x = -\frac{P}{2}$ with the P distance between F and $D \equiv$ direction straight line

IV-5 <u>Polar Coordinate</u>

Let consider the pole of the straight line D perpendicular on the semi axes opposite to polar axle.



From $\frac{\|\vec{PF}\|}{\|PQ\|} = e$ we can deduce $\frac{\vec{PO}}{\vec{PQ}} = e$,

$$\overline{PO} = \rho \text{ and } \overline{PO} = e\,\overline{PQ}$$

$\overline{OP} = \rho$ = radius = Modulus of polar coordinate

$$\overline{PQ} = \overline{QO} + \overline{OP'}$$

$$\cos\theta = \frac{\overline{OP'}}{\overline{OP}} = \frac{\overline{OP'}}{\rho} \Rightarrow OP' = \rho\cos\theta$$

$$\frac{\overline{PO}}{\overline{PQ}} = e \Rightarrow \overline{PO} = e\,\overline{PQ} = e$$

Let p be the distance from pole to the straight line D opposite on polar axe.

So, $\overline{QO} = p \Rightarrow \overline{PQ} = \overline{QO} + \overline{OP'} = p + \rho\cos\theta$

$$\frac{\overline{PO}}{\overline{PQ}} = e \Rightarrow \overline{PO} = e(\overline{PQ}) = \rho = e(p+\rho\cos\theta)$$

$$\rho = ep + e\cos\theta = \rho - e\rho\cos\theta = ep$$

$$\Rightarrow \rho(1-e\cos\theta)$$

$$\rho = \frac{ep}{1-e\cos\theta} \quad \text{or} \quad \rho = \frac{ep}{1+e\cos\theta}$$

The polar equation of conics $\rho = \sqrt{x^2+y^2}$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

The cartesian coordinate ρ will be distance between focus and the corresponding direction line.

$$\boxed{r = \frac{e\rho}{1-e\cos\theta}}$$

If $e=1$: parabola
$e>1$: Hyperbola
$0<e<1$ - Ellipse.

we can change polar coordinate to cartesian coordinate and vice-versa.

Ex: let consider the following conics.

ⓐ $r = \frac{12}{4+3\cos\theta}$ ⓒ $r = \frac{4}{2-3\cos\theta}$ ⓔ $r = \frac{2}{1-\cos\theta}$

ⓑ $r = \frac{16}{5-3\cos\theta}$ ⓓ $r = \frac{8}{3-5\cos\theta}$

Determine the nature of conics and the distance from pole to direction line.

(ii) Give the canonical form if the pole correspond to origine of axis
Give the canonical form if the pole doesn't correspond to the origine of axis)

② let consider the following conics

ⓐ $\frac{x^2}{16} + \frac{y^2}{9} = 1$ ⓑ $\frac{x^2}{9} - \frac{y^2}{4} = 1$

ⓒ $\frac{x^2}{9} + y^2 = 1$

Scanned with CamScanner

Give the ~~form~~ polar form eccentricity and the direction line.

Solution

$$r = \frac{12}{4+3\cos\theta} \Rightarrow \frac{12}{4\left(1+\frac{3}{4}\cos\theta\right)} = \frac{3}{1+\frac{3}{4}\cos\theta}.$$

$$\Rightarrow r = \frac{ep}{1+e\cos\theta}, \quad e = \frac{3}{4}//$$

$0 < e < 1$: $e = \frac{3}{4} < 1$: Ellipse.

$ep = 3 \quad p = \frac{3}{e} = \frac{3}{3/4} = 4//$

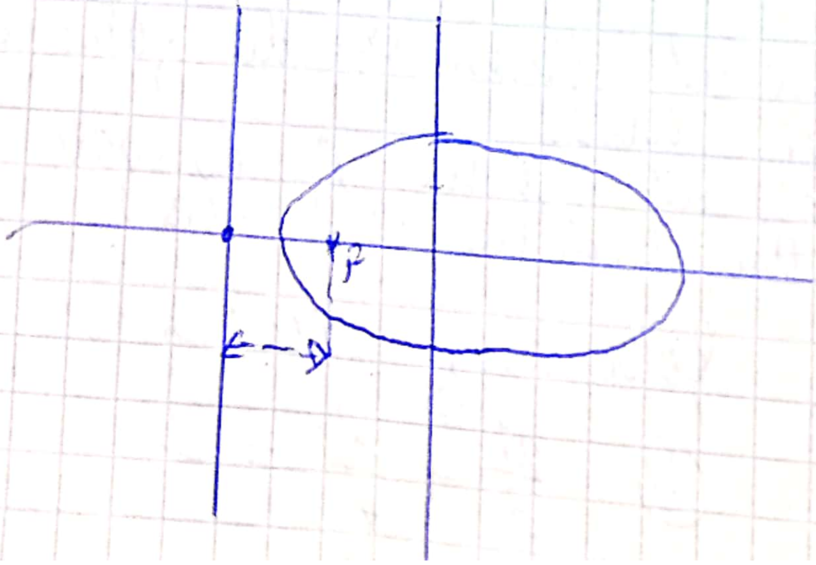
Distance from pole to direction is $p = 4$

$e = \frac{c}{a} = \frac{3}{4}//$ and $x = \frac{a^2}{c}$

$\frac{c}{a} = \frac{3}{4} \Rightarrow \boxed{c = \frac{3}{4}a}$ One distance btn foci and direction line is

$b^2 = a^2 - c^2$

$D = \frac{a^2}{c} = x.$



$$p = \sqrt{\left(c - \frac{a^2}{c}\right)^2 - (0-0)^2}$$

$$p = \left|c - \frac{a^2}{c}\right| \Rightarrow 4 = \frac{c^2 - a^2}{c} = 4c = |c^2 - a^2|$$

$$\begin{cases} 4c = c^2 - a^2 \\ c = \frac{3}{4}a \end{cases} \Rightarrow 4 \cdot \frac{3}{4}a = \left(\frac{3a}{4}\right)^2 - a^2$$

$$3a = \frac{9a^2}{16} - a^2 \Rightarrow 48a = 9a^2 - 16a^2$$

$$48a = -7a^2 \Rightarrow a = \frac{48}{7}$$

$$c = \frac{3}{4} \times \frac{48}{7} = \frac{3}{7}$$

$$b^2 = a^2 - c^2 = \left(\frac{48}{7}\right)^2 - \left(\frac{36}{7}\right)^2$$

$$= \frac{2304}{49} - \frac{1296}{49} = \frac{144}{7}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$r = \frac{12}{4 + 3\cos\theta} \qquad r = \sqrt{x^2 + y^2}, \quad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\sqrt{x^2 + y^2} = \frac{12}{4 + \frac{3x}{\sqrt{x^2 + y^2}}} \Rightarrow \sqrt{x^2 + y^2} = \frac{12\sqrt{x^2 + y^2}}{4\sqrt{x^2 + y^2} + 3x}$$

$$4\sqrt{x^2 + y^2} + 3x = 12$$

$$4\sqrt{x^2 + y^2} = 12 - 3x$$

$$16(x^2 + y^2) = 144 - 72x + 9x^2$$

$$16x^2 - 9x^2 + 72x + 16y^2 = 144$$

$$7x^2 + 72x + 16y^2 = 144$$

$$7\left(x^2 + 2 \cdot \frac{36}{7}\right) + 16y^2 = 144$$

$$7\left(x + \frac{36}{7}\right)^2 - \frac{1296}{7} + 16y^2 = 144$$

$$7\left(x+\frac{36}{7}\right)^2 + 16y^2 = \frac{2304}{7}$$

$$\frac{\left(x+\frac{36}{7}\right)^2}{\frac{2304}{7}} + \frac{y^2}{\frac{144}{7}} = 1 \Rightarrow \frac{\left(x+\frac{36}{7}\right)^2}{\left(\frac{48}{7}\right)^2} + \frac{y^2}{\left(\frac{12}{\sqrt{7}}\right)^2} = 1$$

$$a^2 = \frac{2304}{49} = a = \sqrt{\frac{2304}{49}} = \frac{\sqrt{2304}}{\sqrt{49}} = \frac{48}{7}$$

$$b^2 = \frac{144}{7} \times \frac{7}{7} = \frac{1008}{49}$$

(2) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$a = 4,\ b = 3,\ b^2 = a^2 - c^2$

$c^2 = a^2 - b^2 = 16 - 9 = 7$

$c = \pm\sqrt{7}$

$$e = \frac{c}{a} = \frac{\sqrt{7}}{4}$$

$$p = \left|\frac{a^2}{c} - c\right| = \left|\frac{a^2 - c^2}{c}\right| = \left|\frac{16-7}{\sqrt{7}}\right|$$

$$p = \frac{9\sqrt{7}}{7}$$

$$r = \frac{ep}{1+e\cos\theta} = \frac{\frac{\sqrt{7}}{4} \times \frac{9\sqrt{7}}{7}}{1+\frac{\sqrt{7}}{4}\cos\theta} = \frac{9}{4\frac{1}{4}(4+\sqrt{7}\cos\theta)}$$

$$\boxed{r = \frac{9}{4+\sqrt{7}\cos\theta}}$$

Given parametric coordinates

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \Rightarrow \begin{cases} x = 4\cos t \\ y = 3\sin t \end{cases}$$

IV.6 Conics Reduction

Let $Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0$ be the equation of conics, the coordinate of the centre is obtained by solving the system.

$$\begin{cases} \dfrac{\partial f(x,y)}{\partial x} = 0 \\ \dfrac{\partial f(x,y)}{\partial y} = 0 \end{cases}$$

Let (x_0, y_0) be the centre

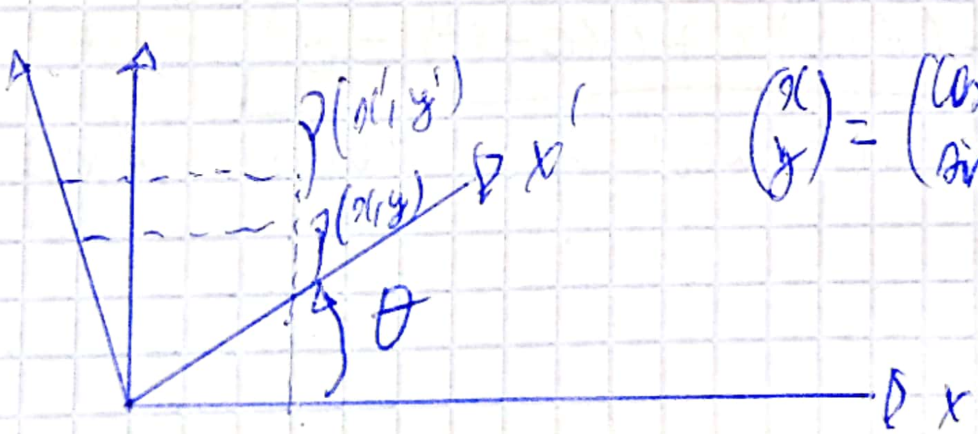
The translation of conics about the centre is obtained by using $\begin{cases} x = x' + x_0 \\ y = y' + y_0 \end{cases}$

The rotation about of axis can be obtained by using

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

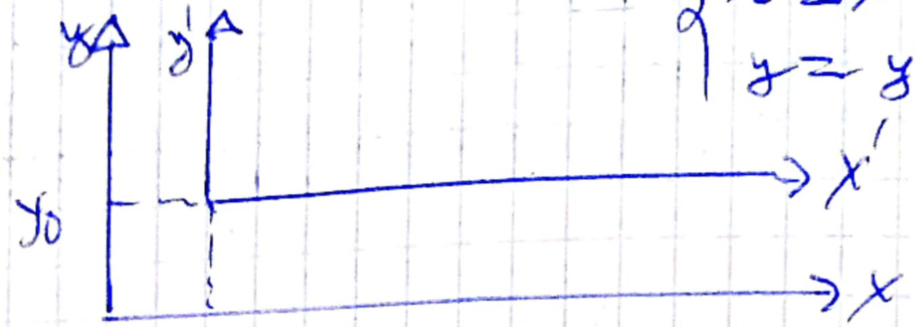
$$X = x'\cos\theta - y'\sin\theta + x_0$$
$$y = y'\sin\theta + y'\cos\theta + y_0$$

where $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation Matrix.



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

For translation.



$$\begin{cases} x = x' + x_0 \\ y = y' + y_0 \end{cases}$$

By using translation ~~the~~ form.

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

becomes.

$$\boxed{Ax'^2 + 2Bx'y' + Cy'^2 + \frac{\Delta}{\delta} = 0}$$

where $\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$ and $\delta = AC - B^2$

Δ and δ are called invariation of conics. By using rotation of (*), the conics is reduced to

$$\boxed{A'x''^2 + B'y''^2 + \frac{\Delta}{\delta} = 0}$$

The coefficient A' and B' are obtained by solving the quadratic equation $\boxed{\lambda^2 - (A+C)\lambda + \delta = 0}$

$$\boxed{\lambda^2 - (A+C)\lambda + (AC - B^2) = 0}$$

The canonical form will be $A'x''^2 + B'y'' + \frac{\Delta}{\delta} = 0$

$\delta > 0$ and $\Delta < 0$: Ellipse
$\Delta > 0$: Empty ellipse
$\Delta = 0$: point ellipse

$\delta < 0$ and $\Delta \neq 0$: Hyperbola
$\Delta = 0$, Hyperbola reduced on its asymptote.

Application

By using conics reduction, give the canonical form of

ⓐ $25x^2 - 14xy + 25y^2 + 64x - 64y - 224 = 0$
ⓑ $7x^2 + 6xy - y^2 + 28x + 12y + 28 = 0$
ⓒ $50x^2 - 8xy + 35y^2 + 100x - 8y + 67 = 0$
ⓓ $41x^2 + 24xy + 34y^2 + 34x - 112y + 129 = 0$

Solution

ⓐ $25x^2 - 14xy + 25y^2 + 64x - 64y - 224 = 0$

$A = 25$ $\quad C = 25$ $\quad E = -32$
$B = -7$ $\quad D = 32$ $\quad F = -224$

$$\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = \begin{vmatrix} 25 & -7 & 32 \\ -7 & 25 & -32 \\ 32 & -32 & -224 \end{vmatrix} \begin{matrix} 25 & -7 \\ -7 & 25 \\ 32 & -32 \end{matrix}$$

$\delta = AC - B^2 = 25^2 - (-7)^2 = 625 - 49$
$= 576$

By the reduction of the conics becomes

$$25x'^2 - 14x'y' + 25y'^2 + \frac{165888}{576} = 0$$

$$25x'^2 - 14x'y' + 25y'^2 - 288 = 0$$

Let solve the quadratic equation

$$x^2 - (A+C)x + \delta = 0$$

$$x^2 - 50x + 576 = 0$$

$$\Delta = (50)^2 - 4(1)(576) = 2500 - 2304 = 196$$
$$= 14^2$$

$$x_1 = \frac{50+14}{2} = 32 \qquad x_2 = \frac{50-14}{2} = 18$$

The conic reduced to

$$A'x''^2 + B'y''^2 + \frac{\Delta}{\delta} = 0$$

$$18x''^2 + 32y''^2 - 288 = 0$$

$$\Rightarrow 18x''^2 + 32y''^2 = 288$$

$$\frac{x''^2}{\frac{288}{18}} + \frac{y''^2}{\frac{288}{32}} = 1 \Rightarrow \boxed{\frac{x''^2}{16} + \frac{y''^2}{9} = 1}$$

Scanned with CamScanner

$$\Rightarrow \begin{cases} y = \frac{b}{a}(x-\alpha)+\beta \\ y = -\frac{b}{a}(x-\alpha)+\beta \end{cases}$$ asymptote,

Exercises

(1) Determine the equation of the ellipse E which is centered on A(1,2) of the focus (6,2) and where the point (4,6) belongs to the ellipse

Answers

- the ellipse is centered on A(1,2) its equation is

$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

- (4,6) belongs to E $\Rightarrow \frac{(4-1)^2}{a^2} + \frac{(6-2)^2}{b^2} = 1$

$$\Rightarrow \frac{9}{a^2} + \frac{16}{b^2} = 1$$

- the semi focus distance is $c = \overline{AF}$ (center - focus)

A(1,2) and F(6,2)

$$\Rightarrow c = \sqrt{(6-1)^2 - (2-2)^2} = \sqrt{25} = 5$$

Because we have E $\Rightarrow b^2 = a^2 - c^2$

Hence $b^2 = a^2 - 25$

thus we have to solve $\begin{cases} \frac{9}{a^2} + \frac{16}{b^2} = 1 & (2) \\ b^2 = a^2 - 25 & (1) \end{cases}$

(1) into (2) gives $\frac{9}{a^2} + \frac{16}{a^2-25} = 1$

$$\frac{9(a^2-25)+16a^2}{a^2(a^2-25)} = 1 \Rightarrow 9a^2 - 9\times 25 + 16a^2 = a^4 - 25a^2$$

$$a^4 - 50a^2 + 9\cdot 25 = 0$$

$$\Delta' = 25^2 - 9\cdot 25 = 625 - 225 = 400 = 20^2$$

$$a^2 = \frac{25 \pm 20}{1} \begin{cases} 45 \\ 5 \end{cases}$$

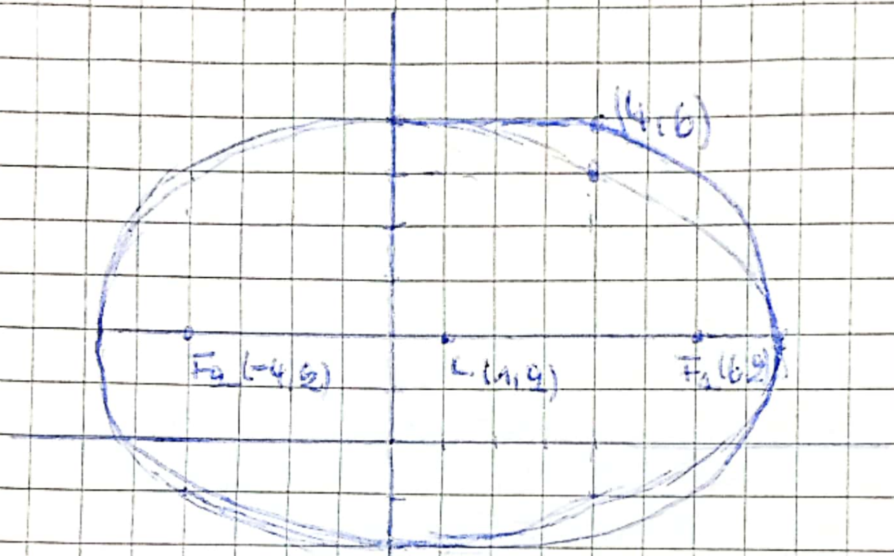
- on the ellipse $a > c$

$a^2 = 45 \Rightarrow a = \sqrt{45} > 5$ ~~rejected~~ taken
$a^2 = 5 \Rightarrow a = \sqrt{5} > 5$ ~~taken~~ rejected

Hence $a^2 = 45 \Rightarrow b^2 = a^2 - c^2 \Rightarrow b^2 = 45 - 25 = 20$

then the equation become $\boxed{\dfrac{(x-1)^2}{45} + \dfrac{(y-2)^2}{20} = 1}$

other focus is $F_2(1-5, 2) \Rightarrow F_2(-4, 2)$



the vertices are $S_1(1+\sqrt{45}, 2)$ and $S_2(1-\sqrt{45}, 2)$
$S_3(1, 2+\sqrt{20})$ and $S_4(1, 2-\sqrt{20})$

(2) Let $9x^2 - 16y^2 - 36x - 32y - 124 = 0$ be the equation of Hyperbola
a/ Determine the coordinate of the center
b/ Determine the coordinate of the focus
c/ Determine the coordinate of the vertices
d/ Determine the equation of asymptotes

<u>Answer</u>
a/ $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases} \Rightarrow \begin{cases} 18x - 36 = 0 \\ -32y - 32 = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = -1 \end{cases}$

$C(2, -1)$

b/ $(9x^2 - 36x) - (16y^2 + 32y) - 124 = 0$
$9(x^2 - 4x) - 16(y^2 + 2y) - 124 = 0$

$9[(x-2)^2-4] + 16[(y+1)^2 - 1] - 124 = 0$

$9(x-2)^2 - 36 + 16(y+1)^2 + 16 - 124 = 0$

$9(x-2)^2 + 16(y+1)^2 = 124 + 36 - 16$

$9(x-2)^2 - 16(y+1)^2 = 144$

$$\boxed{\frac{(x-2)^2}{16} - \frac{(y+1)^2}{9} = 1}$$

In the hyperbola $b^2 = c^2 - a^2 \Rightarrow c^2 = b^2 + a^2$

$c^2 = 16 + 9 = 25 \Rightarrow c = 5$

the focus are $\boxed{\begin{array}{l} F_1(2+5, -1) = F_1(7, -1) \\ F_2(2-5, -1) = F_2(-3, -1) \end{array}}$

c) vertices $\boxed{\begin{array}{l} S_1(2+4, -1) = S_1(6, -1) \\ S_2(2-4, -1) = S_2(-2, -1) \end{array}}$

d) the equation of asymptotes $y - \beta = \pm \frac{b}{a}(x - \alpha)$

$y + 1 = \pm \frac{3}{4}(x-2)$

• $y = \frac{3}{4}(x-2) - 1 \Rightarrow y = \frac{3x}{4} - \frac{3}{2} - 1 \Rightarrow y = \frac{3x}{4} - \frac{5}{2}$

$\Rightarrow \boxed{4y - 3x + 10 = 0}$

• $y = -\frac{3}{4}(x-2) - 1 \Rightarrow y = -\frac{3}{4}x + \frac{3}{2} - 1$

$\Rightarrow \boxed{4y + 3x - 2 = 0}$

Answer exercise number 2

2) $x^2 - 4y^2 - 6x + 16y - 11 = 0$

$A = 1 \quad C = -4 \quad E = 8$

$B = 0 \quad D = -3 \quad F = -11$

$\delta = AC - B^2 = 1(-4) - 0 = -4 < 0$ Hyperbola

$$\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 0 & -4 & 8 \\ -3 & 8 & -11 \end{vmatrix} = 1\begin{vmatrix} -4 & 8 \\ 8 & -11 \end{vmatrix} - 3\begin{vmatrix} 0 & -4 \\ -3 & 8 \end{vmatrix}$$

$\Delta = (44-64) - 3(0-12) = -20 + 36 = 16$

$\delta < 0$ and $\Delta \neq 0 \Rightarrow$ the hyperbola in did

b) canonical equation $A' x''^2 + C' y''^2 + \frac{\Delta}{\delta} = 0$

as $\frac{\Delta}{\delta} = \frac{16}{-4} = -4$

· the coefficients A' and C' are obtained after solving $x^2 - (A+C)x + \delta = 0$

$x^2 + 3x - 4 = 0$

$\Delta' = 9 + 16 = 25 \quad x_1 = \frac{-3+5}{2} = 1, \; x_2 = \frac{-3-5}{2} = -4$

$A' = 1 \quad \rightarrow C' = -4$

$A' = -4 \quad \rightarrow C' = 1$

$x''^2 - 4y''^2 - 4 = 0$ or $-4x''^2 + y''^2 - 4 = 0$

$\Rightarrow \boxed{\frac{x''^2}{4} - y''^2 = 1}$ $\boxed{\frac{y''^2}{4} - x''^2 = 1}$

c) the parametric equation

1/ $a = 2$ and $b = 1$

$$\begin{cases} x = \frac{a}{\cos t} \\ y = b \tan t \end{cases} \Rightarrow \begin{cases} x = \frac{2}{\cos t} \\ y = \tan t \end{cases}$$ t is a parameter

2/ $b = 2$ and $a = 1$

$$\begin{cases} x = \frac{a}{\cos t} \\ y = b \tan t \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\cos t} \\ y = 2 \tan t \end{cases}$$