Engineering Mechanics

Lectures 1&2

Review of the three laws of motion and vector algebra

In this course on Engineering Mechanics, we shall be learning about mechanical interaction between bodies. That is we will learn how different bodies apply forces on one another and how they then balance to keep each other in equilibrium. That will be done in the first part of the course. So in the first part we will be dealing with STATICS. In the second part we then go to the motion of particles and see how does the motion of particles get affected when a force is applied on them. We will first deal with single particles and will then move on to describe the motion of rigid bodies.

The basis of all solutions to mechanics problems are the Newton's laws of motion in one form or the other. The laws are:

First law: *A body does not change its state of motion unless acted upon by a force.* This law is based on observations but in addition it also defines an *inertial frame* . By definition an inertial frame is that in which a body does not change its state of motion unless acted upon by a force. For example to a very good approximation a frame fixed in a room is an inertial frame for motion of balls/ objects in that room. On the other hand if you are sitting in a train that is accelerating, you will see that objects outside are changing their speed without any apparent force. Then the motion of objects outside is changing without any force. The train is a non-inertial frame.

Second law: The second law is also part definition and part observation. It gives the force in terms of a quantity called the mass and the acceleration of a particle. It says that a force of magnitude F applied on a particle gives it an acceleration a proportional to the force. In other words

$$F = ma \,, \qquad (1)$$

where m is identified as the inertial mass of the body. So if the same force - applied either by a spring stretched or compressed to the same length - acting on two different particles produces accelerations a_1 and a_2, we can say that

$$m_1 a_1 = m_2 a_2$$

or

$$m_2 = \left(\frac{a_1}{a_2}\right) m_1 \qquad (2)$$

Thus by comparing accelerations of a particle and of a standard mass (unit mass) when the same force is applied on each one them we get the mass of that particle. Thus gives us the definition of mass. It also gives us how to measure the force via the equation F = ma. One Newton (abbreviated as N) of force is that providing an acceleration of $1 m/s^2$ to a standard mass of 1 kg. If you want to feel how much in 1 Newton , hold your palm horizontally and put a hundred gram weight on it; the force that you feel is about 1N.

Of course you cannot always measure the force applied by accelerating objects. For example if you are pushing a wall, how much force you are applying cannot be measured by observing the acceleration of the wall because the wall is not moving. However once we have adopted a measure of force, we can always measure it by comparing the force applied in some other situation.

In the first part of the course i.e. Statics we consider only equilibrium situations. We will therefore not be looking at F = ma but rather at the balance of different forces applied on a system. In the second part - Dynamics - we will be applying F = ma extensively.

Third Law: Newton's third law states that if a body *A* applies a force F on body *B* , then *B* also applies an equal and opposite force on *A* . (Forces do not cancel such other as they are acting on two different objects)

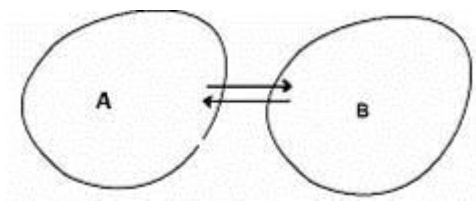


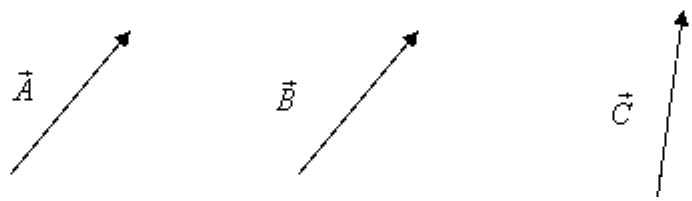
Figure 1

Thus if they start from the position of rest *A* and *B* will tend to move in opposite directions. You may ask: if *A* and *B* are experiencing equal and opposite force, why do they not cancel each other? This is because - as stated above - the forces are acting on two different objects. We shall be using this law a lot both in static as well as in dynamics.

After this preliminary introduction to what we will be doing in the coming lectures, we begin with a review of vectors because the quantities like force, velocities are all vectors and we should therefore know how to work with the vectors. I am sure you have learnt some basic manipulations with vectors in your 12th grade so this lectures is essentially to recapitulate on what you have learnt and also introduce you to one or two new concepts.

You have learnt in the past is that vectors are quantity which have both a magnitude and a direction in contrast to scalar quantities that are specified by their magnitude only. Thus a quantity like force is a vector quantity because when I tell someone that I am applying X- amount of force, by itself it is not meaningful unless I also specify in which direction I am applying this force. Similarly when I ask you where your friend's house is you can't just tell me that it is some 500 meters far. You will also have to tell me that it is 500 meters to the north or 300 meters to the east and four hundred meters to the north from here. Without formally realizing it, you are telling me a about a vector quantity. Thus quantities like displacement, velocity, acceleration, force are vectors. On the other hand the quantities distance, speed and energy are scalar quantities. In the following we discuss the algebra involving vector quantities. We begin with a discussion of the equality of vectors.

Equality of Vectors: Since a vector is defined by the direction and magnitude, two vectors are equal if they have the same magnitude and direction. Thus in figure 2 vector $\vec{A}$ is equal to vector $\vec{B}$ and but not equal to vector $\vec{C}$ although all of them have the same magnitude.



Vectors $\vec{A}$ and $\vec{B}$ are equal to each other but not equal to vector $\vec{C}$

Figure 2

Thus we conclude that any two vectors which have the same magnitude and are parallel to each other are equal. If they are not parallel then they cannot be equal no matter what their magnitude.

In physical situations even two equal vectors may produce different effects depending on where they are located. For example take the force $\vec{F}$ applied on a disc. If applied on the rim it rotates the wheel at a speed different from when it is applied to a point nearer to the center. Thus although it is the same force, applied at different points it produces different effects. On the other hand, imagine a thin rope wrapped on a wheel and being pulled out horizontally from the top. On the rope no matter where the force is applied, the effect is the same. Similarly we may push the wheel by applying the same force at thee end of a stick with same result (see figure 3).

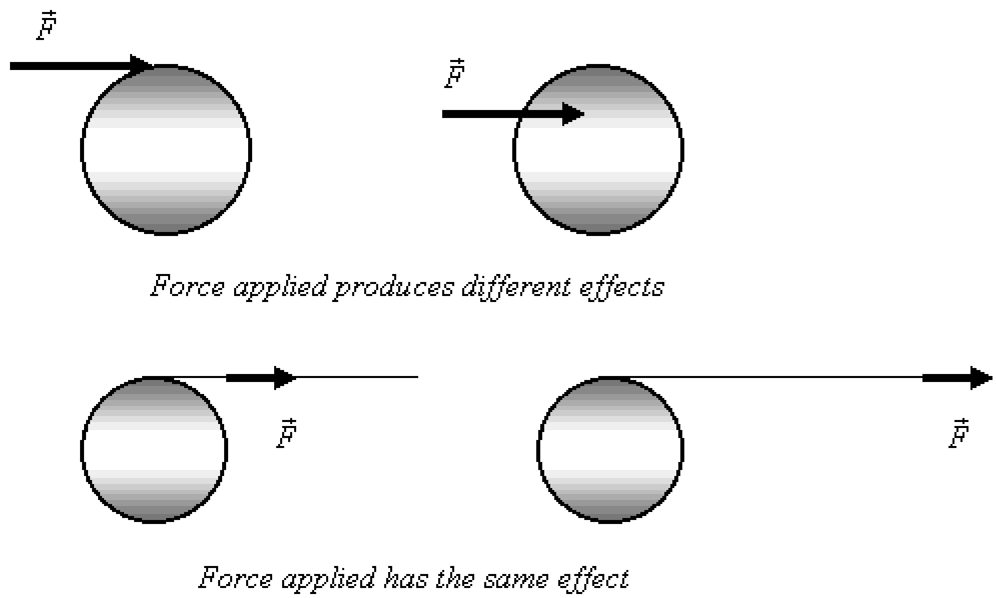
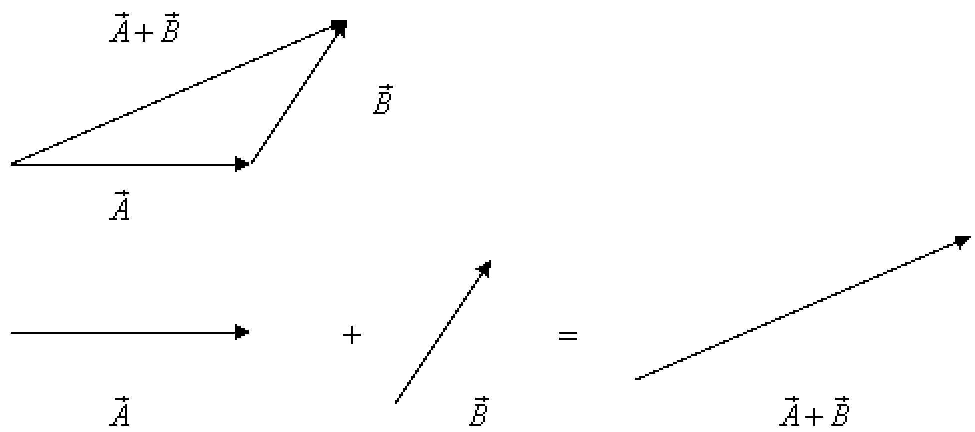


Figure 3

Thus we observe that a force applied anywhere along its line of applications produces the same effect. This is known as transmissibility of force. On the other hand if the same force is applied at a point away from its line of application, the effect produced is different. Thus the transmissibility does not mean that force can be applied anywhere to produce the *same* effect but only at any point on its line of application.

Adding and subtracting two vectors (Graphical Method): When we add two vectors $\vec{A}$ and $\vec{B}$ by graphical method to get $\vec{A}+\vec{B}$, we take vector $\vec{A}$, put the tail of $\vec{B}$ on the head of $\vec{A}$.Then we draw a vector from the tail of $\vec{A}$ to the head of $\vec{B}$. That vector represents the resultant $\vec{A}+\vec{B}$ (Figure 4). I leave it as an exercise for you to show that $\vec{A}+\vec{B}=\vec{B}+\vec{A}$. In other words, show that vector addition is commutative.



Adding two vectors

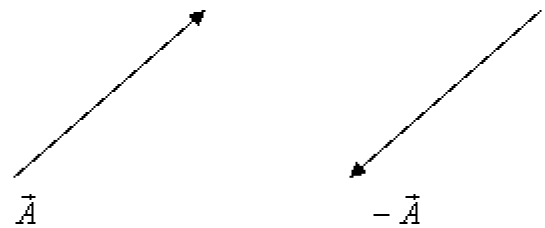
Figure 4

Let us try to understand that it is indeed meaningful to add two vectors like this. Imagine the following situations. Suppose when we hit a ball, we can give it velocity $\vec{B}$. Now imagine a ball is moving with velocity $\vec{A}$ and you hit it an additional velocity $\vec{B}$. From experience you know that the ball will now start moving in a direction different from that of $\vec{A}$. This final direction is the direction of $\left(\vec{A}+\vec{B}\right)$ and the magnitude of velocity now is going to be given by the length of $\left(\vec{A}+\vec{B}\right)$.

Now if we add a vector $\vec{A}$ to itself, it is clear from the graphical method that its magnitude is going to be 2 times the magnitude of $\vec{A}$ and the direction is going to remain the same as that of $\vec{A}$. This is equivalent to multiplying the vector $\vec{A}$ by 2. Similarly if 3 vectors are added we get the resultant $3\vec{A}$. So we have now got the idea of multiplying a vector by a number *n* . If simply means: add the vector *n* times and this results in giving a vector in the same direction with a magnitude that *n* times larger.

You may now ask: can I multiply by a negative number? The answer is yes. Let us see what happens, for example, when I multiply a vector $\vec{A}$ by -1. Recall from your school mathematics that multiplying by -1 changes the number to the other side of the number line. Thus the number -2 is two steps to the left of 0 whereas the number 2 is two steps to the right. It is exactly the same with vectors. If $\vec{A}$ represents a vector to the right, $-\vec{A}$ would represent a vector in the direction opposite i.e. to the left. It is now easy to understand what does the vector $-\vec{A}$ represent? It is a vector of the same magnitude as that of $\vec{A}$

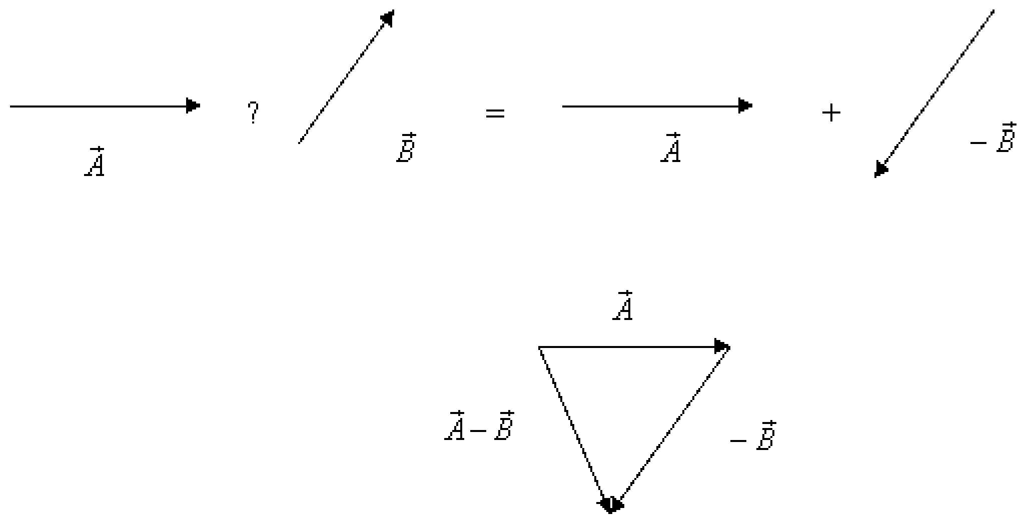
but in the direction opposite to it (Figure 5). Having defined $-\vec{A}$, it is now easy to see what is the vector $-m\vec{A}$? It is a vector of magnitude $m|\vec{A}|$ in the direction opposite to $\vec{A}$.



A vector and its negative

Figure 5

Having defined $-\vec{A}$, it is now straightforward to subtract one vector from the other. To subtract a vector $\vec{B}$ from $\vec{A}$, we simply add $-\vec{B}$ to $\vec{A}$ that is $\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$. Thus to subtract vector $\vec{B}$ from $\vec{A}$ graphically, we add $\vec{A}$ and $-\vec{B}$. This is shown in figure 6.



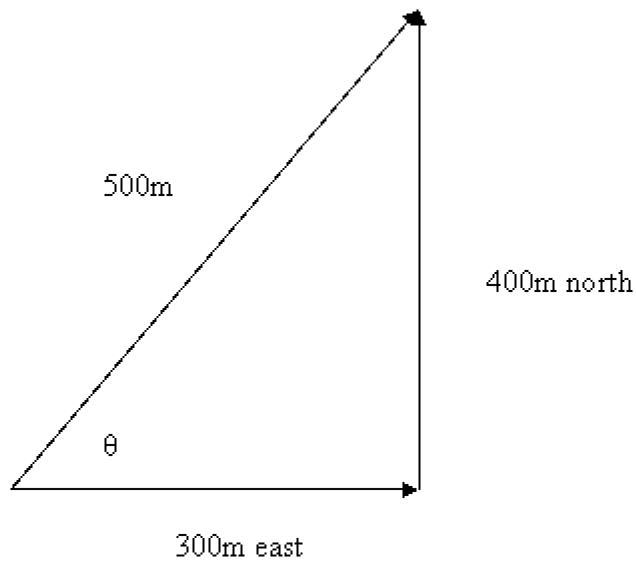
Subtracting vector $\vec{B}$ from vector $\vec{A}$

Figure 6

Again I leave it as an exercise for you to show that $\left(\vec{A}-\vec{B}\right)$ is not equal to $\left(\vec{B}-\vec{A}\right)$ but $\left(\vec{B}-\vec{A}\right)$ = - $\left(\vec{A}-\vec{B}\right)$. We now solve a couple of examples.

Example1: A person walks 300m to the east and 400m to the north to reach his friend's house. What is the total displacement of the person, and what is the total distance traveled by him?

Recall that distance is a scalar quantity. Thus the total distance covered is 700m. Displacement, on the other hand, is a vector quantity so to find the net displacement, we add the two vectors to get a displacement of 500m at an angle $\theta = \tan^{-1}\left(\frac{4}{3}\right)$ from east to north (Figure 7).



Adding displacements of 300m east and 400m north

Figure 7

Example 2 : Two persons are pushing a box so that the net force on the box is 12N to the east If one of the person is applying a force 5N to the north, what is the force applied by the other person.

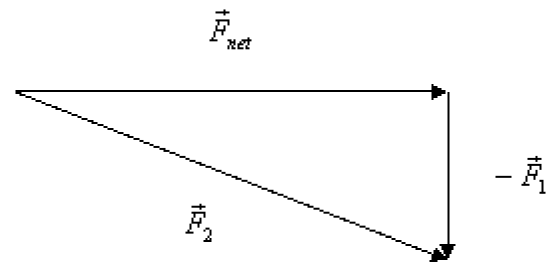
Let the force by person applying 5N be denoted by $\vec{F}_1$ and that by the other person by $\vec{F}_2$. We then have

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$$

so that

$$\vec{F}_2 = \vec{F}_{net} - \vec{F}_1$$

Solution for $\vec{F}_2$ is given graphically in figure 8. The force comes out to be 13N at an angle of $\theta = \tan^{-1}\left(\frac{5}{12}\right)$ from east to south.



Finding force applied by a person when the net force and that applied by one of the persons is given.

Figure 8

Although graphical way is nice to visualize vectors in two dimensions, it becomes difficult to work with it in three dimensions, and also when many vectors and many operations with them are involved. So vector algebra is best done by representing them in terms of their components along the x, y & z axes in space. We now discuss how to this is done.

To represent vectors in terms of their x,y and z components, let us first introduce the concept of unit vector. A unit vector $\hat{n}$ in a particular direction is a vector of magnitude '1' in that direction. So a vector in that particular direction can be written as a number times the unit vector $\hat{n}$. Let us denote the unit vector in x-direction as $\hat{i}$, in y-direction as $\hat{j}$ and in z-direction as $\hat{k}$. Now any vector can be described as a sum of three vectors $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ in the directions x, y and z, respectively, in any order (recall that order does not matter because vector sum is commutative). Then a vector

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

Further, using the concept of unit vectors, we can write $\vec{A}_x = A_x\hat{i}$, where A_x is a number. Similarly $\vec{A}_y = A_y\hat{j}$ and $\vec{A}_z = A_z\hat{k}$. So the vector above can be written as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

where A_x, A_y and A_z are known as the x, y, & z components of the vector. For example a vector $\vec{A} = 3\hat{i} + 3\hat{j} + 4\hat{k}$ would look as shown in figure 9.

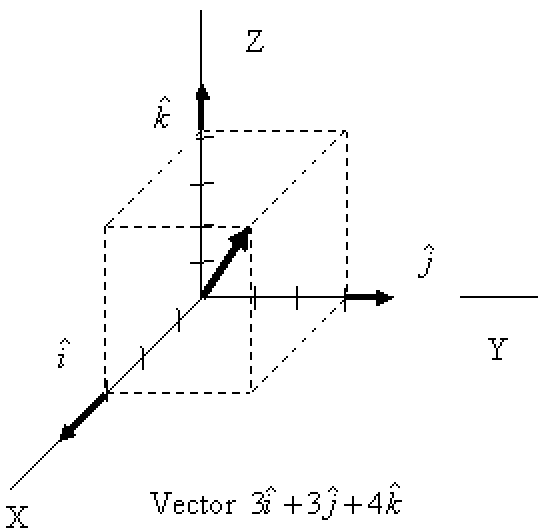


Figure 9

It is clear from figure 9 that the magnitude of the vector $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ is $\left|\vec{A}\right|=\sqrt{A_x^2+A_y^2+A_z^2}$. Now when we add two vector, say $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$, all we have to do is to add their x-components, y-components and the z-components and then combine them to get

$$\vec{A}+\vec{B}=(A_x+B_x)\hat{i}+(A_y+B_y)\hat{j}+(A_z+B_z)\hat{k}$$

Similarly multiplying a vector by a number is same as increasing all its components by the same amount. Thus

$$m\vec{A}=(mA_x)\hat{i}+(mA_y)\hat{j}+(mA_z)\hat{k}$$

How about the multiplying by -1? It just changes the sign of all the components. Putting it all together we see that

$$\vec{A}-\vec{B}=(A_x-B_x)\hat{i}+(A_y-B_y)\hat{j}+(A_z-B_z)\hat{k}$$

Having done the addition and subtraction of two vectors, we now want to look at the product of two vectors. Let us see what all possible products do we get when we multiply components of two vectors. By multiplying all components with one another, we have in all nine numbers shown below:

$$\begin{pmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{pmatrix}$$

The question is how do we define the product of two vectors from the nine different numbers obtained above? We will delay the answer for some time and come back to this question after we establish the transformation properties of scalars and vectors. By transformation properties we mean how does a scalar quantity or the components of a vector quantity change when we look at them from a different (rotated) frame?

Let us first look at a scalar quantity. As an example, we take the distance traveled by a person. If we say that the distance covered by a person in going from one place to another is 1000m in one frame, it remains the same irrespective of whether we look at it from the frame (xy) or in a frame ($x' y'$) rotates about the z-axis (see figure 10).

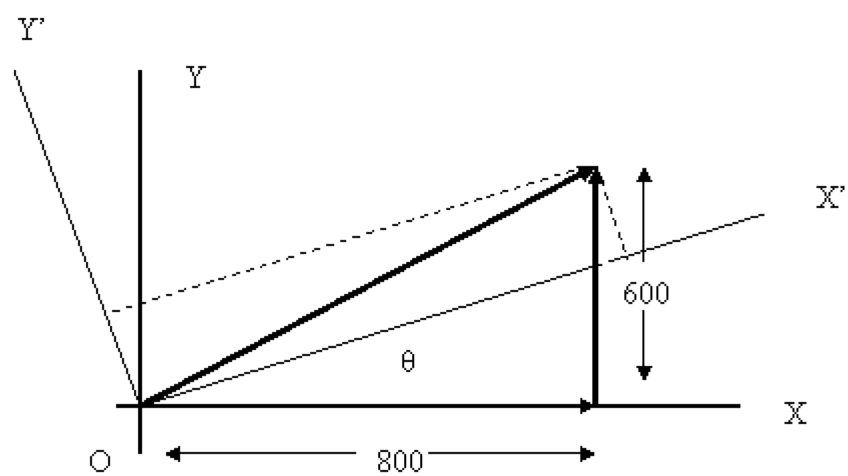


Figure 10

Let us now say that a person moves 800 meter along the x-axis and 600 meters along the y-axis so that his net displacement is a vector of 1000m in magnitude at an angle of from the x-axis as shown in figure 10. The total distance traveled by the person is 1400m. Now let us look at the same situation frame different frame which has its x' & y' axis rotated about the z- axis. Note that the total distance traveled by the person (a scalar quantity) remains the same, 1400m, in both the frames. Further, whereas the magnitude of the displacement & its direction in space remains unchanged, its components along the x' and y' axis, shown by dashed lines in figure 10, are now different. Thus we conclude the scalar quantity remains unchanged when seen from a rotational frame. The component of a given vector are however different in the rotated frame, as demonstrated by the example above. Let us now see how the components in the original frame and the rotated frame are related.

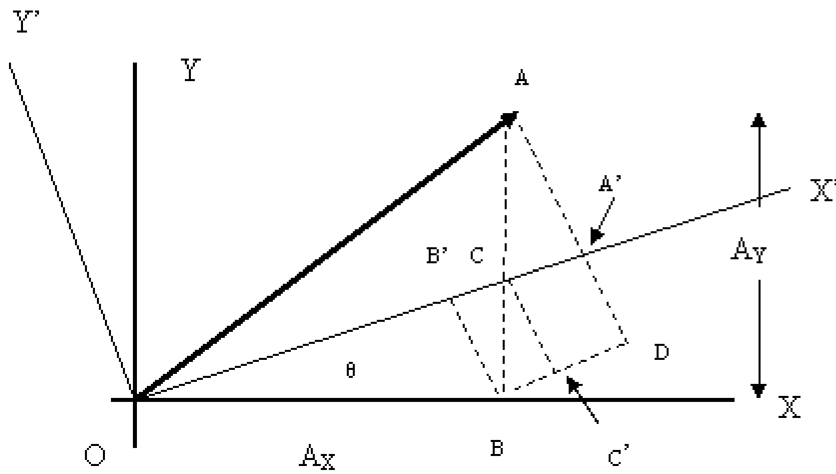


Figure 11

In figure 11, OA is a vector with A_x = OB , A_y = AB, $A_{x'}$ = OA' and $A_{y'}$ = AA'. Using the dashed lines drawn in the figure, we obtain

$$
\begin{aligned}
A_{x'} = OA' &= OB' + B'A' \\
&= OB\cos\theta + B'C + C'D \\
&= OB\cos\theta + B'C + CA' \\
&= OB\cos\theta + BC\sin\theta + CA\sin\theta \\
&= A_x\cos\theta + (BC + CA)\sin\theta \\
&= A_x\cos\theta + A_y\sin\theta
\end{aligned}
$$

Similarly

$$
\begin{aligned}
A_{y'} = AA' &= AD - A'D \\
&= AD - B'B \\
&= BA\cos\theta - OB\sin\theta. \\
&= -A_x\sin\theta + A_y\cos\theta
\end{aligned}
$$

So we learn that if the same vector is observed from a frame obtained by a rotation about the z-axis by an angle θ, its x and y components in the new frame are

$$
\begin{aligned}
A_x' &= A_x\cos\theta + A_y\sin\theta \\
A_y' &= -A_x\sin\theta + A_y\cos\theta \\
A_z' &= A_z
\end{aligned}
$$

One can similarly define how components mix when rotation is about the y or the x- axis. Under the y axis rotation

$$A_z' = A_z \cos\theta + A_y \sin\theta$$
$$A_x' = -A_z \sin\theta + A_y \cos\theta$$

And under a rotation about the x-axis

$$A_y' = A_y \cos\theta + A_z \sin\theta$$
$$A_z' = -A_y \sin\theta + A_z \cos\theta$$

Let us summarize the results obtained above:

1. Scalar quantity is specified by a number and that number remains the same in two different frames rotated with respect to each other.

2. A vector quantity is specified by its components along the x, y, and the z axes and when seen from another frame rotated with respect to a given frame, these components change according to the rules derived above.

We are now ready to get back to defining the product of two vectors. Recall that we had a collection of nine quantities:

$$\begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

We are now going to mix these quantities in such a manner that one combination will give a scalar quantity whereas the other one will give us a vector quantity. This then defines the scalar and vector product of two vectors.

Scalar or dot product: Now it is easy to show that $\left(A_x B_x + A_y B_y + A_z B_z\right)$ **is a scalar quantity. To show this we calculate this quantity in a rotated frame (rotation could be about the x, y or the z axis) that is obtain** $\left(A_x' B_x' + A_y' B_y' + A_z' B_z'\right)$ **and show that it is equal to** $\left(A_x B_x + A_y B_y + A_z B_z\right)$**. As an example we show it for a frame rotated about the z-axis with respect to the other one. In this case**

$$A_x' = \cos A_x + \sin\theta A_y \quad A_y' = -\sin\theta A_x + \cos\theta A_y$$
$$B_x' = \cos B_x + \sin\theta B_y \quad B_y' = -\sin B_x + \cos\theta B_y$$
$$A_z' = A_z, B_z' = B_z$$

Therefore we get

$$A'_x B'_x + A'_y B'_y + A'_z B'_z = \left(\cos\theta A_x + \sin\theta A_y\right)\left(\cos\theta B_x + \sin\theta B_y\right)$$
$$+\left(-\sin\theta A_x + \cos\theta A_y\right)\left(-\sin\theta B_x + \cos\theta B_y\right) + A_z B_z$$

$$= \left(\cos^2\theta + \sin^2\theta\right)\left(A_x B_x\right) + \left(\cos^2\theta + \sin^2\theta\right)\left(A_y B_y\right) + A_z B_z$$

$$= A_x B_x + A_y B_y + A_z B_z$$

One can similarly show it for rotations about other axes, which is left as an exercise. This then leads us to define the scalar product of two vectors $\vec{A}$ and $\vec{B}$ as

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

As shown above this value remain unchanged when view from two different frame-one rotated with respect to the other. Thus it is a scalar quantity and this product is known as the scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$. It is straightforward to see from the definition above that the dot product is commutative that is $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.

Scalar product of two vectors can also be written in another form involving the magnitudes of these vectors and the angle between them as

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

where $\left|\vec{A}\right|$ and $\left|\vec{B}\right|$ are the magnitudes of the two vectors, and θ is the angle between them. Notice that although $\left|\vec{A}\right|$ and $\left|\vec{B}\right| > 0, \vec{A}\cdot\vec{B}$ can be negative or positive depending on the angle between them. Further, if two non-zero vectors are perpendicular, $\vec{A}\cdot\vec{B} = 0$. From the formula above, it is also apparent that if we take vector $\vec{B}$ to be a unit vector, the dot product $\vec{A}\cdot\vec{B}$ represents the component of $\vec{A}$ in the direction of $\vec{B}$. Thus the scalar product between two vectors is the product of the magnitude of one vector with the magnitude of the component of the other vector in its direction. Try to see it pictorially yourself. We also write the dot products of the unit vectors along the x, y, and the z axes. These are $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

Vector or cross product: In defining the scalar product above, we have used three out of the nine possible products of the components of two vectors. From the six of these that are left i.e. $A_x B_y, B_x A_y, A_x B_z, B_x A_z, A_y B_z,$ and $B_y A_z$, if we define the vector

$$\vec{A}\times\vec{B} = (A_y B_z - A_z B_x)\hat{i} + (A_z B_y - A_z B_x)\hat{j} + (A_x B_y - A_y B_z)\hat{k}$$

This is known as the vector or cross product of the two vectors. By calling this expression a vector, we implicitly mean that its component transform like those of a vector. Let us again take the example of looking at the components of this quantity from two frames rotated with respect to each other about the z-axis. In that case the x component of the vector product in the rotated frame is

$$\begin{aligned}\left(\vec{A}\times\vec{B}\right)_{x'} &= A_{y'}B_{z'} - B_{y'}A_{z'}\\ &= \left(-A_x \sin\theta + A_y\cos\theta\right)B_z - A_z(-B_x\sin\theta + B_y\cos\theta)\\ &= (A_yB_z - B_yA_z)\cos\theta + (A_zB_x - B_zA_x)\sin\theta\\ &= (\vec{A}\times\vec{B})_x\cos\theta + (\vec{A}\times\vec{B})_y\sin\theta\end{aligned}$$

and the y component is

$$\begin{aligned}\left(\vec{A}\times\vec{B}\right)_{y'} &= A_{z'}B_{x'} - B_{z'}A_{x'}\\ &= A_z\left(B_x\cos\theta + B_y\sin\theta\right) - B_z(A_x\cos\theta + A_y\sin\theta)\\ &= -\sin\theta(A_yB_z - B_yA_z) + \cos\theta(A_zB_x - B_zA_x)\\ &= -\left(\vec{A}\times\vec{B}\right)_x\sin\theta + \left(\vec{A}\times\vec{B}\right)_y\cos\theta\end{aligned}$$

Thus we see that the components of the vector product defined above do indeed transform like those of a vector. We leave it as an exercise to show that when the other frame is obtained by rotating about the x and the y axes also, the transformation of the components is like that of a vector. This is known as the vector or the cross product of vectors $\vec{A}$ and $\vec{B}$. It can also be written in the form of a determinant as

$$\vec{A}\times\vec{B} = \begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{pmatrix}_s$$

Notice that this is the only contribution that transforms in this manner. For example

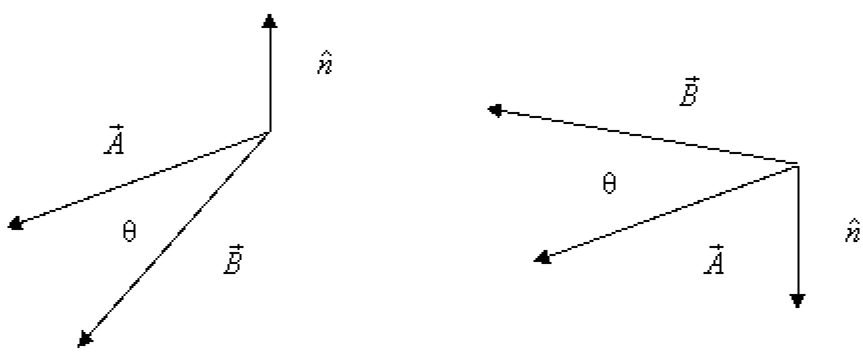
$$(A_yB_z + A_zB_y)\hat{\mathrm{i}} + \hat{\mathrm{j}}(A_zB_x + A_xB_z) + \hat{\mathrm{k}}(A_xB_y + A_yB_x)$$

does not transform like a vector; I leave it as an exercise for you to show. So this cannot form a vector.

Now if we take the dot product of $\vec{A}$ or $\vec{B}$ with $\vec{A}\times\vec{B}$, the result is zero as is easy to see. This implies that the vector product of two vectors is perpendicular to both of them. As such an alternate expression for the vector product of $\vec{A}$ and $\vec{B}$ is

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector in the direction perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ in such a way that if the fingers of the right hand turn from $\vec{A}$ to $\vec{B}$ through the smaller of the angle between them, the thumb gives the direction of in direction of $\hat{n}$. It is also clear from this expression that the vector product of two non-zero vectors will vanish if the vectors are parallel i.e. the angle between them is zero.

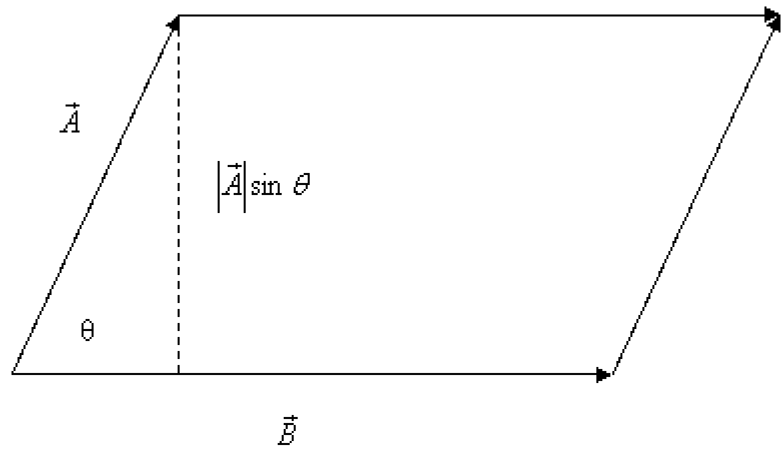


Direction of $\vec{A}\times\vec{B}$ for two different orientations of $\vec{A}$ and $\vec{B}$

Figure 12

The vector product between two vectors is not commutative in that $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$ but rather $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

Geometric interpretation of cross product : The magnitude of the cross-product $(\vec{A}\times\vec{B})$, which is equal to $|\vec{A}||\vec{B}|\sin\theta$, is the area of a parallelogram formed by vectors $\vec{A}\,\&\,\vec{B}$. This is shown in figure 13.



Area of a parallelogram formed by two vectors is equal to the magnitude of their cross product

Figure 13

Derivative of a vector: After reviewing the vector algebra, we would now like to introduce you to the idea of differentiating a vector quantity. Here we take a vector $\vec{A}(t)$ as depending on one parameter, say time *t* , and evaluate the derivative $\frac{d\vec{A}(t)}{dt}$. This is similar to what we do for a regular function. We evaluate the vector $\vec{A}(t+\Delta t)$ at time *(t+ Δt)* , subtract $\vec{A}(t)$ from it, divide the difference $\Delta\vec{A}(t)$ by *Δt* and then take the limit *Δ t → 0* . This is shown in figure 14. Thus

$$\frac{d\vec{A}(t)}{dt} = \lim\ \Delta t \to 0 \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

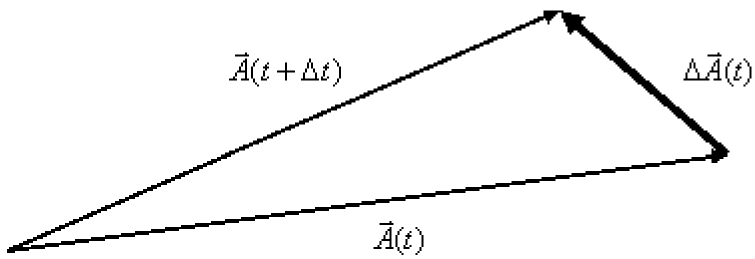


Figure 14

The derivative is easily understood if we think in terms of its derivatives. If we write a vector as

$$\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}$$

then the derivative of the vector is given as

$$\frac{d\vec{A}(t)}{dt} = \frac{dA_x(t)}{dt}\hat{i} + \frac{dA_y(t)}{dt}\hat{j} + \frac{dA_z(t)}{dt}\hat{k}$$

Notice that only the components are differentiated, because the unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ are fixed in space and therefore do not change with time. Later when we learn about polar coordinates, we will encounter unit vectors which also change with time. In that case when taking derivative of a vector, the components as well as the unit vectors both have to be differentiated.

Using the definition above, it is easy to show that in differentiating the product of two vectors, the usual chain rule can be applied. This gives

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

and

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

This pretty much sums up our introduction to vectors. I leave this lecture by giving you three exercises.

1. Show that $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{A}\times\vec{C})$ and that $\vec{A}\cdot(\vec{B}\times\vec{C})$ is the volume of a parallelepiped formed by $\vec{A},\vec{B}\ \&\ \vec{C}$.

2. Show that $\vec{A}\cdot(\vec{B}\times\vec{C})$ can also be written as the determinant

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

1. Show that if the magnitude of a vector quantity $\vec{A}(t)$ is a fixed, its derivative with respect to *t* will be perpendicular to it. Can you think of an everyday example of this?

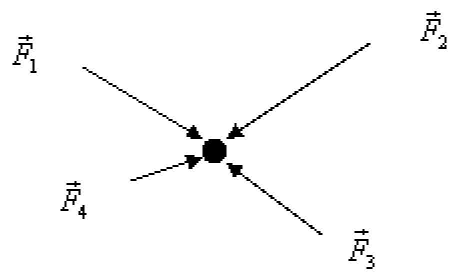
Lecture 3
Equilibrium of bodies I

In the previous lecture, we discussed three laws of motion and reviewed some basic aspects of vector algebra. We are now going to apply these to understand equilibrium of bodies. In the static part when we say that a body is in equilibrium, what we mean is that the body is not moving at all even though there may be forces acting on it. (In general equilibrium means that there is no acceleration i.e., the body is moving with constant velocity but in this special case we take this constant to be zero).

Let us start by observing what all can a force do to a body? One obvious thing it does is to accelerate a body. So if we take a point particle P and apply a force on it, it will accelerate. Thus if we want its acceleration to be zero, the sum of all forces applied on it must vanish. This is the condition for equilibrium of a point particle. So for a point particle the equilibrium condition is

$$\sum_i \vec{F}_i$$

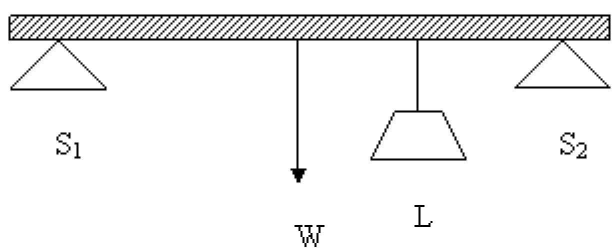
where $\vec{F}_i; i = 1,2,3....$ are the forces applied on the point particle (see figure 13)



A particle in equilibrium under four forces

Figure 1

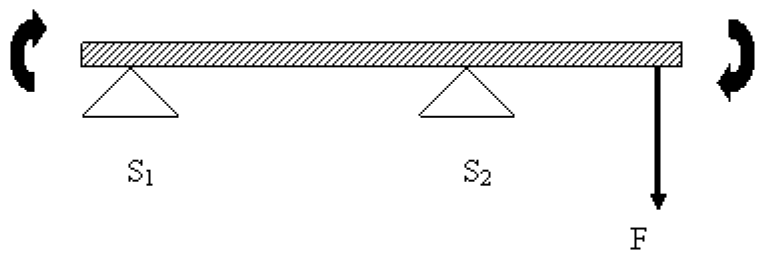
That is all there is to the equilibrium of a point particle. But in engineering problems we deal not with point particles but with extended objects. An example is a beam holding a load as shown in figure 2. The beam is equilibrium under its own weight *W* , the load *L* and the forces that the supports S_1 and S_2 apply on it.



A beam of weight W on supports S_1 and S_2 and holding a load L

Figure 2

To consider equilibrium of such extended bodies, we need to see the other effects that a force produces on them. In these bodies, in addition to providing acceleration to the body, an applied force has two more effects. One it tends to rotate the body and two it deforms the body. Thus a beam put on two supports S_1 and S_2 tends to rotate clockwise about S_2 when a force F is applied downwards (figure 3).



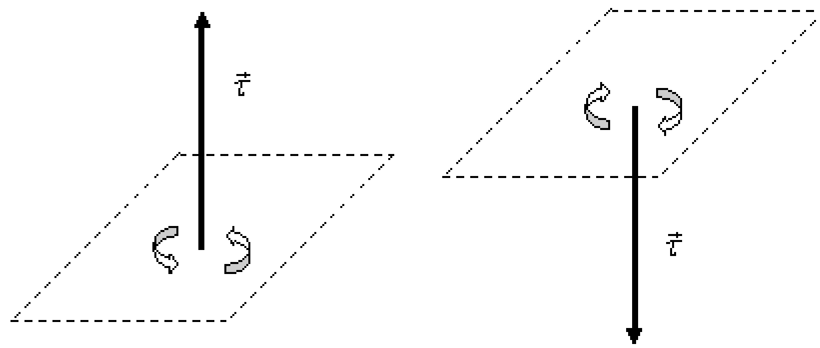
A beam on supports S_1 and S_2 tends to rotate clockwise under the force F

Figure 3

The strength or ability of a force $\vec{F}$ to rotate the body about a point *O* is given by the torque $\vec{\tau}$ generated by it. The torque is defined as the vector product of the displacement vector $\vec{r}_O$ from *O* to the point where the force is applied. Thus

$$\vec{\tau} = \vec{r}_O \times \vec{F}$$

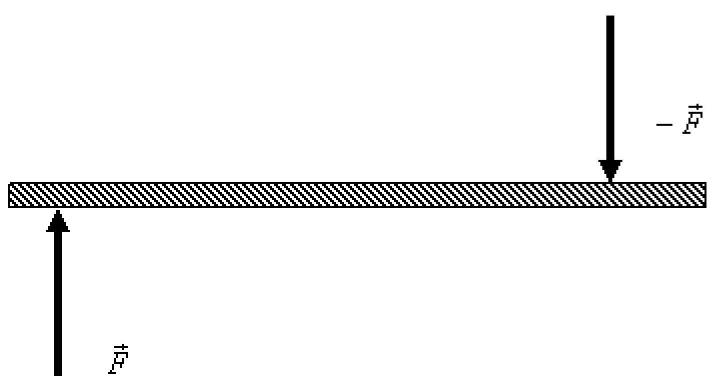
This is also known as the moment of the force. Thus in figure 3 above, the torque about S_2 will be given by the distance from the support times the force and its direction will be into the plane of the paper. From the way that the torque is defined, the torque in a given direction tends to rotate the body on which it is applied in the plane perpendicular to the direction of the torque. Further, the direction of rotation is obtained by aligning the thumb of one's right hand with the direction of the torque; the fingers then show the way that the body tends to rotate (see figure 4). Notice that the torque due to a force will vanish if the force $\vec{F}$ is parallel to $\vec{r}$.



Sense of rotation for a given direction of torque

Figure 4

We now make a subtle point about the tendency of force to rotate a body. It is that even if the net force applied on a body is zero, the torque generated by them may vanish i.e. the forces will not give any acceleration to the body but would tend to rotate it. For example if we apply equal and opposite forces at two ends of a rod, as shown in figure 5, the net force is zero but the rod still has a tendency to rotate. So in considering equilibrium of bodies, we not only have to make sure that the net force is zero but can also that the net torque is also zero.



The net force on the rod is zero but the torque is not

Figure 5

A third possibility of the action by a force, which we have ignored above, and which is highly explicit in the case of a mass on top of a spring, is that the force also deforms bodies. Thus in the case of a beam under a force, the beam may deform in various ways: it may get compressed, it may get elongated or may bend. A load on top of a spring obviously deforms it by a large amount. In the first case we assume the deformation to be small and therefore negligible i.e., we assume that the internal forces are so strong that they adjust so that there is no deformation by the applied external force. This is known as treating the body as a rigid body. In this course, we are going to assume that all bodies are rigid. So the third kind of action is not considered at all.

So now focus strictly on the equilibrium of rigid bodies: As stated, we are going to assume that internal forces are so great that the body does not deform. The only conditions for equilibrium in them are:

(1) The body should not accelerate/ should not move which, as discussed earlier, is ensured if $\sum_i \vec{F}_i = 0$ that is the sum of all forces acting on it must be zero no matter at what points on the body they are applied. For example consider the beam in figure 2. Let the forces applied by the supports S_1 and S_2 be F_1 and F_2, respectively. Then for equilibrium, it is required that

$$\vec{F}_1 + \vec{F}_2 + \vec{W} + \vec{L} = 0$$

Assuming the direction towards the top of the page to be y-direction, this translates to

$$F_1\hat{j} + F_2\hat{j} - W\hat{j} - L\hat{j} = 0 \text{ or } F_1 + F_2 - W - L = 0$$

The condition is sufficient to make sure that the net force on the rod is zero. But as we learned earlier, and also our everyday experience tells us that even a zero net force can give rise to a turning of the rod. So F_1 and F_2 must be applied at such points that the net torque on the beam is also zero. This is given below as the second rule for equilibrium.

(2) Summation of moment of forces about any point in the body is zero i.e. $\sum_i \vec{\tau}_{iO} = 0$, where $\vec{\tau}_{iO}$ is the torque due to the force $\vec{F}_i$ about point *O*. One may ask at this point whether $\sum_i \vec{\tau}_{iO} = 0$ should be taken about many different points or is it sufficient to take it about any one convenient point. The answer is that any one convenient point is sufficient because if condition (1) above is satisfied, i.e. net force on the body is zero then the torque as is independent of point about which it is taken. We will prove it later.

These two conditions are both necessary and sufficient condition for equilibrium. That is all we need to do to achieve equilibrium so in principle solving for equilibrium is quite easy and what we should learn is how to apply these condition efficiently in different engineering situations. We are therefore going to spend time on these topics individually.

We start with a few simple examples:

Example 1: A person is holding a 100N weight (that is roughly a 10kg mass) by a light weight (negligible mass) rod AB. The rod is 1.5m long and weight is hanging at a distance of 1m from the end A, which is on a table (see figure 6). How much force should the person apply to hold the weight?

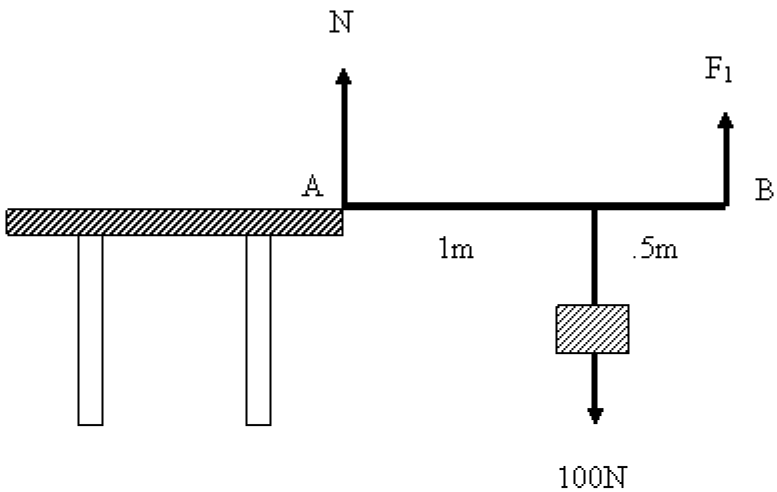


Figure 6

Let the normal reaction of the table on the rod be N and the force by the point be F_1. Then the two equilibrium conditions give

$$\sum \vec{F} = 0 \Rightarrow (F_1 + N - 100)\hat{j} = 0 \Rightarrow F_1 + N = 100 \qquad (1)$$

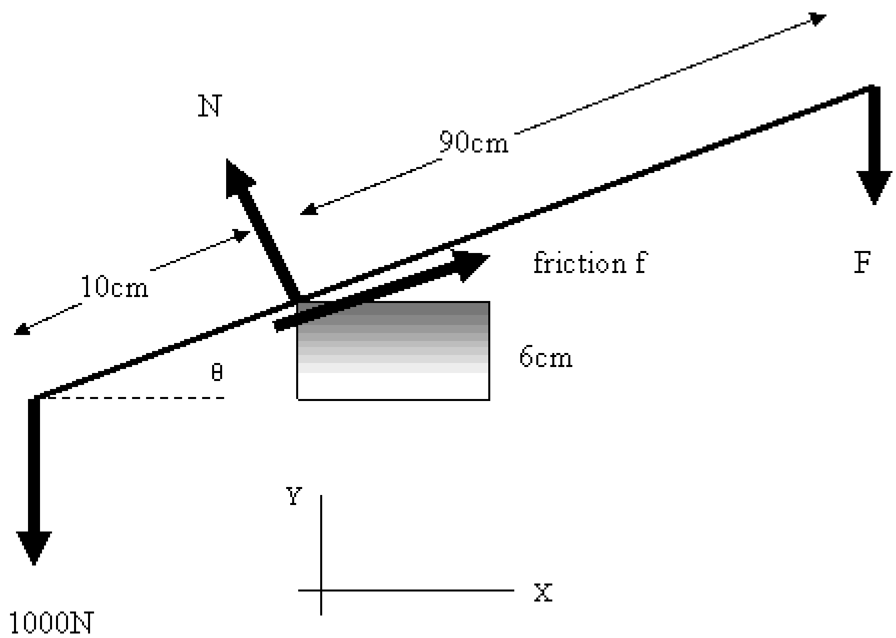
$$\sum \vec{\tau}_{iA} = 0 \Rightarrow \hat{i} \times -100\hat{j} + 1.5\hat{i} \times F_1\hat{j} = 0 \qquad (2)$$

$$-100\hat{k} + F_1 \times 1.5\hat{k} = 0$$

$$\text{or } \; 1.5F_1 = 100 \quad \Rightarrow \quad F_1 = \frac{100}{1.5} = \left(\frac{200}{3}\right)\text{N}$$

$$\text{and } \; N = 100 - F_1 = 100 - \frac{200}{3} = \frac{100}{3}\text{N}$$

Example 2: As the second illustration we take the example of a lever that you may have used sometime or the other. We are trying to lift a 1000N (~100kg mass) weight by putting a light weight but strong rod as shown in the figure using the edge of a brick as the fulcrum. The height of the brick is 6cm. The question we ask is: what is the value of the force applied in the vertical direction that is needed to lift the weight? Assume the brick corner to be rough so that it provides frictional force.



(Note: If the brick did not provide friction, the force applied cannot be only in the vertical direction as that would not be sufficient to cancel the horizontal component of N). Let us see what happens if the brick offered no friction and we applied a force in the vertical direction. The fulcrum applies a force *N* perpendicular to the rod so if we apply only a vertical force, the rod will tend to slip to the left because of the component of *N* in that direction. Try it out on a smooth corner and see that it does happen. However, if the friction is there then the rod will not slip. Let us apply the equilibrium conditions in such a situation. The balance of forces gives

$$\sum \vec{F} = 0 \Rightarrow F(N\sin\theta - f\cos\theta)\hat{i} + (N\cos\theta - f\sin\theta - 1000 - F)\hat{j} = 0$$
$$or \;\; N\sin\theta = f\cos\theta$$
$$N\cos\theta + f\sin\theta - F - 1000 = 0$$

Let us choose the fulcrum as the point about which we balance the torque. It gives

Then

$$\sum \vec{\tau} = 0 \Rightarrow 0.9\hat{r} \times -F\hat{j} + (-0.1)\hat{r} \times -1000\hat{j} = 0$$
$$\Rightarrow \left(-\cdot 9\cos\theta F + 100\cos\theta\right)\hat{k} = 0$$
$$\text{or } F = 111\cdot 11\,\text{N}$$

The normal force and the frictional force can now be calculated with the other two equations obtained above by the force balance equation.

In the example above, we have calculated the torques and have also used normal force applied on a surface. We are going to encounter these quantities again and again in solving engineering problems. So let us study each one of them in detail.

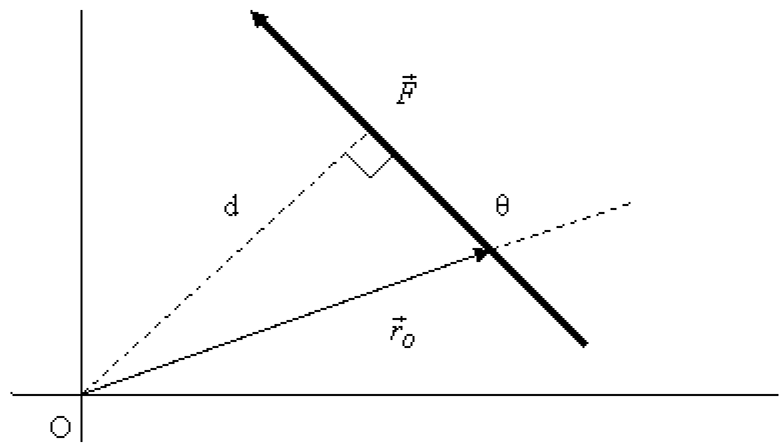
Torque due to a force: As discussed earlier, torque about a point due to a force $\vec{F}$ is obtained as the vector product

$$\vec{\tau}_O = \vec{r}_O \times \vec{F}$$
$$= (yF_z - F_y z)\hat{i} + (zF_x - xF_z)\hat{j} + (xF_y - yF_x)\hat{k}$$

where $\vec{r}_O$ is a vector from the point *O* to the point where the force is being applied. Actually $\vec{r}_O$ could be a vector from *O* to any point along the line of action of the force as we will see below. The magnitude of the torque is given as

$$|\vec{\tau}_O| = |\vec{F}||\vec{r}_O|\sin\theta$$

Thus the magnitude of torque is equal to the product of the magnitude of the force and the perpendicular distance $d = |\vec{r}_O|\sin(180^\circ - \theta) = |\vec{r}_O|\sin\theta$ from *O* to the line of action of the force as shown in figure 7 in the plane containing point *O* and the force vector. Since this distance is fixed, the torque due to a force can be calculated by taking vector $\vec{r}_O$ to be any vector from *O* to the line of action of the force. The unit of a torque is Newton-meter or simply Nm.



Torque is equal to the product of the magnitude of the force and its perpendicular distance d from O

Figure 7

Let us look at an example of this in 2 dimensions.

Example 3: Let there be a force of 20 N applied along the vector going from point (1,2) to point (5,3). So the force can be written as its magnitude times the unit vector from (1,2) to (5,3). Thus

$$\vec{F} = \frac{20\left(4\hat{i} + \vec{j}\right)}{\sqrt{17}}$$

Torque can be calculated about *O* by taking $\vec{r}$ to be either $\left(\hat{i} + 2\hat{j}\right) \, or \, \left(5\hat{i} + 3\hat{j}\right)$. As argued above, the answer should be the same irrespective of which $\vec{r}$ we choose. Let us see that. By taking $\vec{r}$ to be $\left(\hat{i} + 2\hat{j}\right)$ we get

$$\vec{\tau}_O = \frac{\left(\hat{i} + 2\hat{j}\right) \times 20\left(4\hat{i} + \hat{j}\right)}{\sqrt{17}}$$

$$= \frac{20}{\sqrt{17}}(\hat{k} - 8\hat{k}) = -\frac{140\hat{k}}{\sqrt{17}}$$

On the other hand, with $\vec{r} = \left(5\hat{i} + 3\hat{j}\right)$ we get

$$\vec{\tau}_O = \frac{(5\hat{i}+3\hat{j})\times 20(4\hat{i}+\hat{j})}{\sqrt{17}}$$

$$= \frac{20}{\sqrt{17}}(5\hat{k}-12\hat{k}) = -\frac{140\hat{k}}{\sqrt{17}}$$

Which is the same as that obtained with $\vec{r} = (\hat{i}+2\hat{j})$. Thus we see that the torque is the same no matter where along the line of action is the force applied. This is known as the **transmissibility** of the force. So we again write that

$$\vec{\tau}_O = \vec{r}\times\vec{F}$$

where $\vec{r}$ is any vector from the origin to the line of action of the force.

If there are many forces applied on a body then the total moment about *O* is the vector sum of all other moments i.e.

$$\vec{\tau}_O = \sum \vec{r}_{iO}\times\vec{F}_i$$

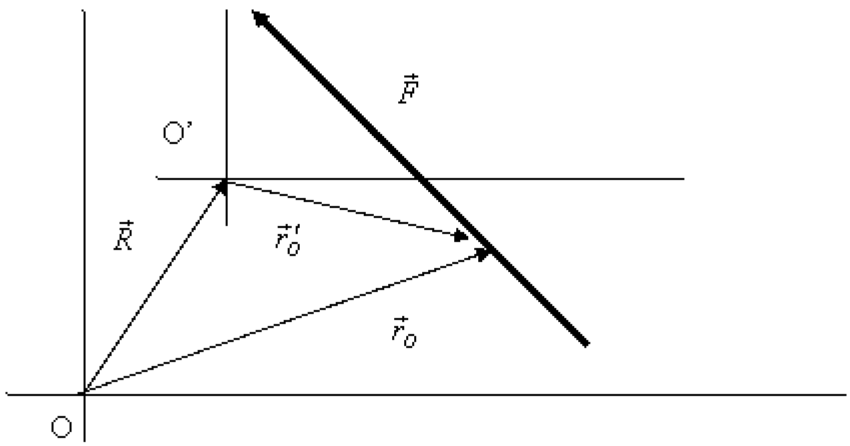
As a special case if the forces are all applied at the same point *j* then

$$\vec{\tau}_O = \sum \vec{r}_{iO}\times\vec{F}_i = \vec{r}_{jO}\times\sum_i \vec{F}_i$$

$$= \vec{r}_{jO}\times\vec{F}_{net}$$

This is known as **Varignon's theorem**. Its usefulness arises from the fact that the torque due to a given force can be calculated as the sum of torques due to its components.

As would be clear to you from the discussion so far torque depends on the location of point *O* . If for the same applied force, the torque is taken about a different point, the torque would come out to be different. However, as mentioned earlier, there is one special case when the torque is independent of the force applied and that is when the net force(vector sum of all forces) on the system is zero. Let us prove that now: Consider the torque of a force being calculated about two different points *O* and *O'* (figure 8).



Torque about two different points O and O' separated by $\vec{R}$

Figure 8

The torques about *O* and *O'* and their difference is:

$$\vec{\tau}_O = \sum \vec{r}_{iO} \times \vec{F}_i \quad \text{and} \quad \vec{\tau}'_O = \sum \vec{r}'_{iO} \times \vec{F}_i$$

$$\Rightarrow \vec{\tau}'_O - \vec{\tau}_O = \sum \left(\vec{r}'_{iO} - \vec{r}_{iO}\right) \times \vec{F}_i$$

But from the figure above

$$\vec{r}'_{iO} - \vec{r}_{iO} = \vec{R}$$

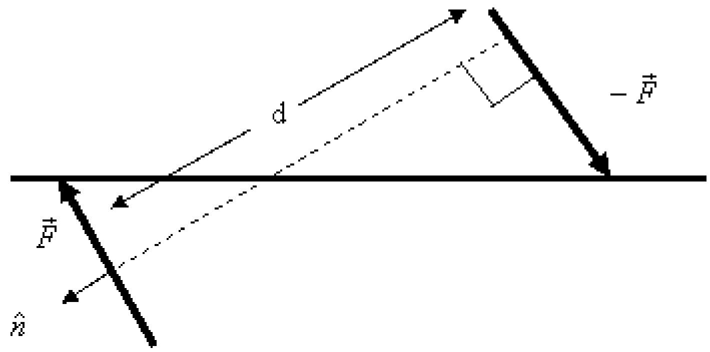
Therefore

$$\vec{\tau}'_O - \vec{\tau}_O = \sum \vec{R} \times \vec{F}_i = \vec{R} \times \sum \vec{F}_i$$

Now if the net force is zero, $\sum \vec{F}_i$ is zero and the difference between the torques about two different points also vanishes. A particular example of the net force being zero is two equal magnitude forces in directions opposite to each other and applied at a distance from one another, as in figure 5 above and also shown in figure 9 below. This is known as a couple and the corresponding torque with respect to any point is given as

$$\vec{\tau}_{couple} = (\hat{n} \times \vec{F}) d.$$

where $\hat{n}$ is a unit vector perpendicular to the forces coming out of the space between them and *d* is the perpendicular distance between the forces (see figure 9).



A couple

Figure 9

Since the net force due to a couple is zero, the only action a couple has on a body is to tend to rotate it. Further the moment of a couple is independent of the origin, and so it can be applied anywhere on the body and it will have the same effect on the body. We can even change the magnitude of the force and alter the distance between them keeping the magnitude of the couple the same. Then also the effect of couple will be the same. Such vectors whose effect remains unchanged irrespective of where they are applied are known as free vectors. Free vectors have a nice property that they can be added irrespective of where they are applied without changing the effect they produce. Thus a couple is a free vector (Is force a free vector?). It is represented by the symbols

Representing a couple

Figure 10

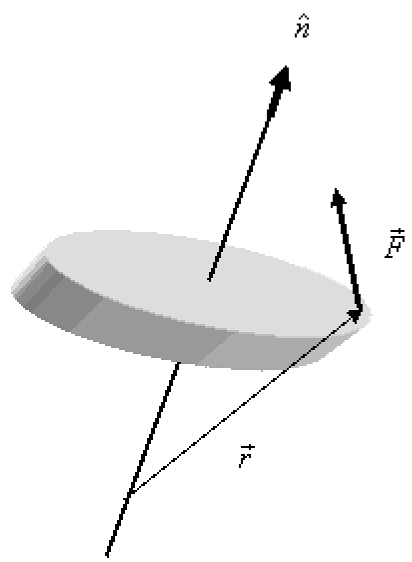
with the arrows clearly giving the sense of rotation. Keep in mind though that the direction of the couple (in the vector sense) is perpendicular to the plane in which the forces forming the couple are.

Next we focus on the moment of a force about an axis.

Moment of force about an axis: So far we have talked about moment of a force about a point only. However, many a times a body rotates about an axis. This is the situations you have bean studying in you 12th grade. For example a disc rotating about an axis fixed in two fixed ball bearings. In this case what affects the rotation is the component of the torque along the axis, where the torque is taken about a point *O* (the point can be chosen arbitrarily) on the axis as given in figure 11. Thus

$$\vec{\tau}_{aboutaxis} = \hat{n} \cdot (\vec{r} \times \vec{F})$$

where $\hat{n}$ is the unit vector along the axis direction and $\vec{r}$ is the vector from point *O* on the axis to the force $\vec{F}$.



Disc experiencing a torque about an axis

Figure 11

Using vector identities (exercise at the end of Lecture 1), it can also be written as

$$\vec{\tau}_{aboutaxis} = (\hat{n} \times \vec{r}) \cdot \vec{F}$$

Thus the moment of a force about an axis is the magnitude of the component of the force in the plane perpendicular to the axis times its perpendicular distance from the axis. Thus if a force is pointing towards the axis, the torque generated by this force about the axis would be zero. This can be understood as follows. When a force is applied, forces are generated at the ends of the axis being held on a one place. These forces together with $\vec{F}$ generate the torque when components along the axis by responsible for rotation of the body about the axis, in the same manner, the couple about the axis is given by the component of the couple moment in the direction for the axis. You can work it out; it is actually equal to the component of the force in the plane perpendicular to the axis times the distance $(\perp)$ of the force line of action from the axis. One point about the moment about an axis, it is independent of the origin since it depends only on the distance $\perp$ of the force the axis.

As an example let us consider a disc of radius 30 cm with its axis along the z-axis and its centre at z=0. Let a force $\vec{F} = (30\hat{i} + 20\hat{j} - 10\hat{k})N$ act on it at the point $(10\hat{i} + 10\hat{j})$ on the disc. We now find its moment about its axis. The axis has $\hat{n} = \hat{k}$. We take the origin at the centre of the disc to calculate

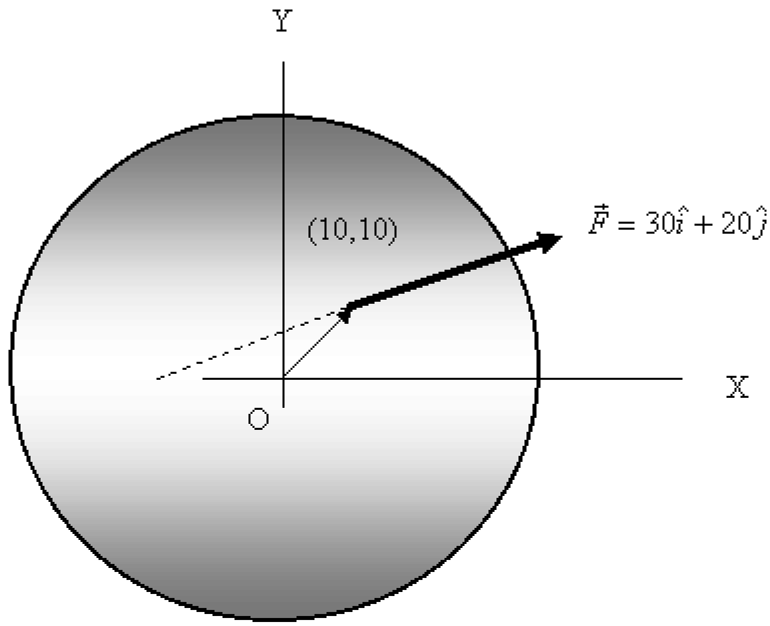
$$\begin{aligned}
\vec{\tau} &= \vec{r} \times \vec{F} \\
&= \left(10\hat{i} + 10\hat{j}\right) \times \left(30\hat{i} + 20\hat{j} - 10\hat{k}\right) \\
&= 200\hat{k} + 100\hat{j} - 300\hat{k} - 100\hat{i} \\
&= -100\hat{i} + 100\hat{j} - 100\hat{k}
\end{aligned}$$

Therefore the torque about the z-axis is

$$\vec{\tau} \cdot \hat{k} = -100\,Nm$$

Thus the torque about the axis is in the negative z direction which means that it would tend to rotate the disc clockwise.

Let us now see if it fits with our conventional way of calculating torque of a force about an axis. For the force $\vec{F} = 30\hat{i} + 20\hat{j} - 10\hat{k}$ the z-component of the force will not give any torque about the z-axis because it cannot rotate the body about the z-axis. So the only component of the force that gives torque about the z-axis is $\left(30\hat{i} + 20\hat{j}\right)$ that acts on the point as shown in figure 12. The magnitude of this force is $10\sqrt{13}$.



Force $\vec{F} = 30\hat{i} + 20\hat{j}$ acting at point $10\hat{i} + 10\hat{j}$ on a disc

Figure 12

The equation of line along which the force acts is

$$(y-10) = \frac{2}{3}(x-10)$$
$$\text{or } 3y = 2x+10$$

To find the perpendicular distance of this line from the origin, we consider a line perpendicular to this line $\left(slope = -\frac{3}{2}\right)$ passing through the origin and consider the point where it intersects with $3y = 2x+10$. The perpendicular line is

$$y = -\frac{3}{2}x \text{ or } 2y = -3x$$

Solving for the intersection point we get

$$x = -\frac{20}{13} \text{ and } y = \frac{30}{13}$$

which gives the perpendicular distance of the line of force from the centre to be

$$d = \frac{10}{\sqrt{13}}$$

Then torque about z-axis therefore is therefore $\tau = |F|.d = 10\sqrt{13} \times \frac{10}{\sqrt{13}} = 100$ clockwise, which is the same as obtained that earlier. I would like you to notice that even in this simple example using vector algebra makes life quite easy.

Let us summarize this lecture by summarizing what we have learnt:

(1) For equilibrium of a body

$$\sum \vec{F} = 0 \text{ and}$$
$$\sum \vec{\tau}_o = 0$$

are necessary and sufficient conditions.

(2) The torque about a point due to forces applied on a body, $\vec{\tau}_o = \sum_i \vec{r}_i \times \vec{F}_i$ is an origin dependent quantity but for special case of $\sum \vec{F}_i = 0$ it is origin independent.

(3) A particular case of $\sum_i \vec{F}_i = 0$ is a couple moment when two forces are equal & opposite and are separated by distance *d* . The couple moment is $|F|d$.

(4) Torque about an axis is given by it component along the axis. Thus y and axis $\hat{n}$ is along direction.

(5) $\tau_{aboutaxis} = \hat{n} \cdot \vec{\tau}$

Lecture 4
Equilibrium of bodies II

In the previous lecture we have defined a couple moment. With this definition, we can now represent a force $\vec{F}$ applied on a body pivoted at a point as the sum of the same force on it at the pivot and a couple acting on it. This is shown in figure 1. Thus if the bar shown in figure 1 is in equilibrium, the pivot must be applying a force $-\vec{F}$ and a counter couple moment on it.

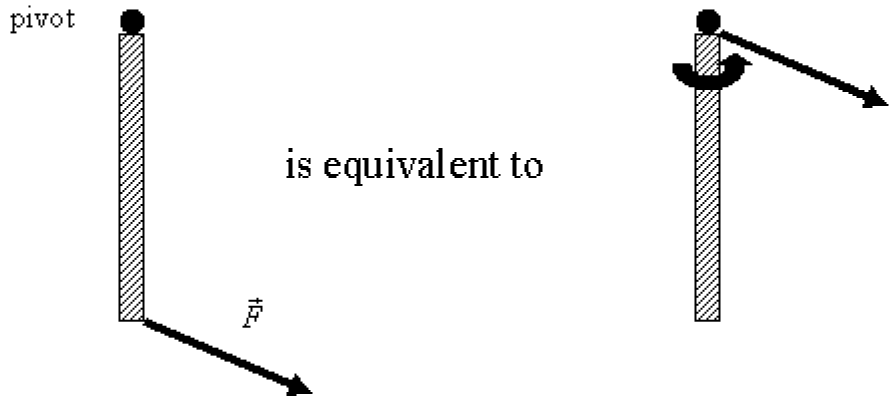


Figure 1

To see the equivalence, let us take the example above and add a zero force to the system at the pivot point. This does not really change the force applied on the system. However, the trick is to take this zero force to be made up of forces $\vec{F}$ and $-\vec{F}$ as shown in figure 2. Now the original force $\vec{F}$ and $-\vec{F}$ at the pivot are separated by distance *d* and therefore form a couple moment of magnitude *Fd* . In addition there is a force $\vec{F}$ on the body at the pivot point. The combination is therefore a force $\vec{F}$ at the pivot point P and a couple moment $\tau = Fd$. Notice that I am not saying τ about the pivot. This is because a couple is a free vector and its effect is the same no matter at which point it is specified.

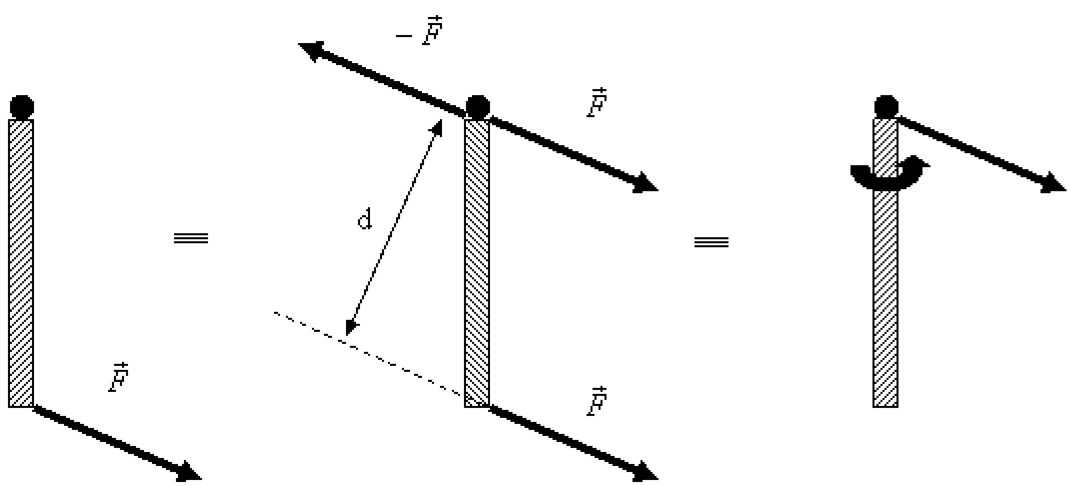


Figure 2

Example: You must have seen the gear shift handle in old buses. It is of Zigzag shape. Let it be of the shape shown in figure 3: 60cm at an angle of 45° from the x-axis, 30cm parallel to x-axis and then 30cm again at 45° from the x-axis, all in the x-y plane shown in figure 3. To change gear a driver applies a force of $\vec{F} = (-5\hat{i} + 5\hat{j} - 2\hat{k})\mathrm{N}$ on the head of the handle. We want to know what is the equivalent force and moment at the bottom i.e., at the origin of the handle.

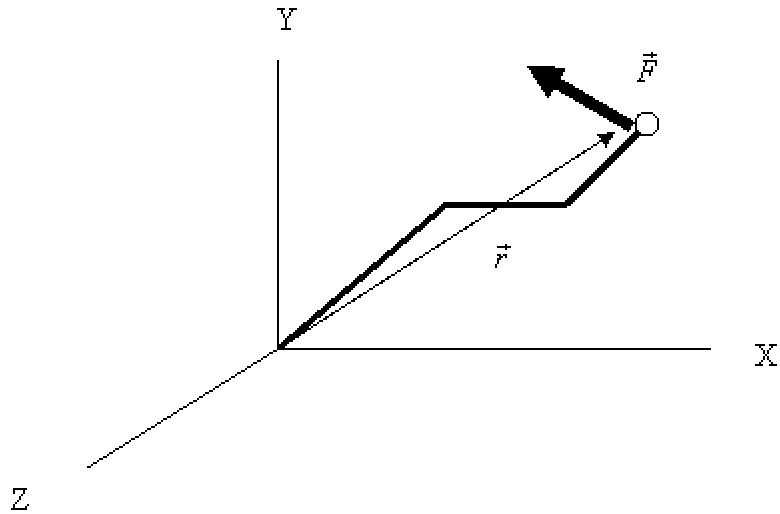


Figure 3

For this again we can apply a zero force i.e., ($\vec{F}$ and $-\vec{F}$) at the bottom so that original force and $-\vec{F}$ give a couple moment

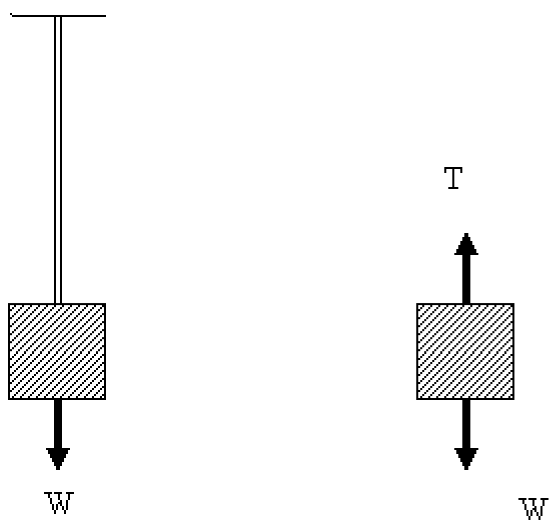
$$
\begin{aligned}
\tau &= \vec{r} \times \vec{F} \\
&= \left(\frac{60}{\sqrt{2}}\hat{i} + \frac{60}{\sqrt{2}}\hat{j} + 30\hat{i} + \frac{30}{\sqrt{2}}\hat{i} + \frac{30}{\sqrt{2}}\hat{j}\right) \times (-5\hat{i} + 5\hat{j} - 2\hat{k}) \\
&= \left(\frac{90 + 30\sqrt{2}}{\sqrt{2}}\hat{i} + \frac{90}{\sqrt{2}}\hat{j}\right) \times (-5\hat{i} + 5\hat{j} - 2\hat{k}) \\
&= \frac{450 + 150\sqrt{2}}{\sqrt{2}}\hat{k} + \frac{180 + 60\sqrt{2}}{\sqrt{2}}\hat{j} + \frac{450}{\sqrt{2}}\hat{k} - 90\sqrt{2}\hat{i} \\
&= -90\sqrt{2}\hat{i} + (90\sqrt{2} + 60)\hat{j} + 600\sqrt{2}\hat{k}
\end{aligned}
$$

Thus equivalent force system is a force $\vec{F} = (-5\hat{i} + 5\hat{j} - 2\hat{k})\mathrm{N}$ at the bottom and a couple equal to $-90\sqrt{2}\hat{i} + (90\sqrt{2} + 60)\hat{j} + 600\sqrt{2}\hat{k}$ Nm.

Having obtained equivalent force systems, next we wish to discuss what kind of forces and moments do different elements used in engineering mechanics apply on other elements.

Forces and couples generated by various elements: As we solve engineering problems, we come across many different elements that are used in engineering structures. We discuss some of them below focusing our attention on what kind of forces and torques do they give rise to.

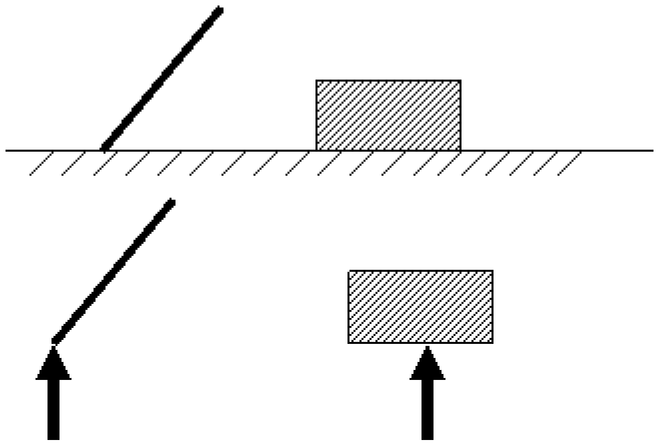
The simplest element is a string that can apply a tension. However a string can only pull by the tension generated in it but not push. For example, a string holding a weight W will develop a tension $T = W$ in it so that the net force on the weight is a tension T pulling the weight up and weight W pulling it down. Thus if the weight is in equilibrium, $T = W$. This is shown in figure 4.



A weight being held in equilibrium by the tension in the string holding it

Figure 4

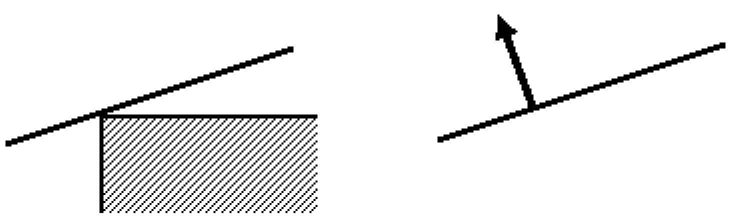
The second kind of force that is applied when two elements come in contact is that applied by a surface. A smooth surface always applies a force normal to itself. The forces on a rod and on a box applied by the surface are shown in figure 5. Thus as far as the equilibrium is concerned, for an object on a smooth surface, the surface is equivalent to a force normal to it.



A smooth surface applies a force normal to it on the objects kept on it

Figure 5

Imagine what would have happened had the force by the surface not been normal. Then an object put on a surface would start moving along the surface because of the component of the force along the surface. By the same argument if there is a smooth surface near an edge, the force on the surface due to the edge (and by Newton's IIIrd law the force on the edge due to the surface) will be normal to the surface. See figure 6.

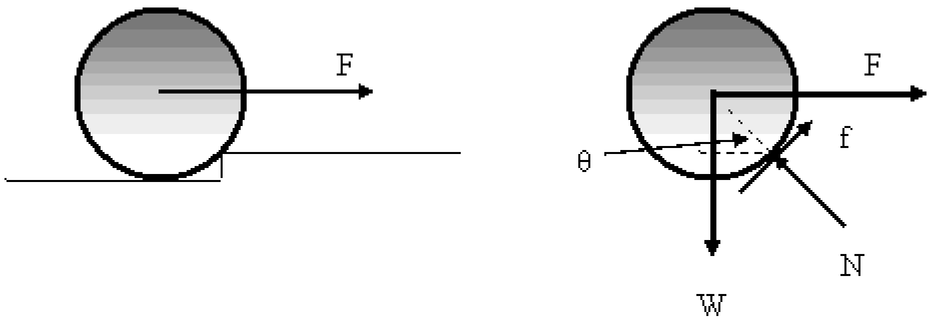


Force applied by an edge on a smooth surface

Figure 6

On the other hand if the surface is not smooth, it is then capable of applying a force along the surface also. This force is due to friction.

Let us now solve the well known example of a roller of radius *r* being pulled over a step as shown in figure 7. The height of the step is *h*. What is minimum force F required if the roller is pulled in the direction shown and is about to roll over the step. What are the normal and frictional forces at that instant?



A roller being pulled over a step and various forces on it when it is about to roll over

Figure 7

When the roller is about to roll over the step, there will be no normal reaction from the lower surface and therefore the roller will be under equilibrium under the influence of its weight *W*, the applied force *F* and the normal reaction *N* and the frictional force *f* applied by the edge of the step. To calculate the force *F*, we apply the torque equation about the edge to get

$$F(r-h) = W\sqrt{2rh-h^2}$$
$$\text{or } F = W\frac{\sqrt{2rh-h^2}}{(r-h)}$$

To find *N* and *f* we apply the force equation

$$\sum \vec{F} = 0$$

That can be written in the component form a

$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

Let us look at these equations.

$\sum F_x = 0$ gives

$- N\cos\theta + f\sin\theta + F = 0$

and

$\sum F_y = 0$ gives

$N\sin\theta + f\cos\theta - W = 0$

with $\cos\theta = \dfrac{\sqrt{2rh - h^2}}{r}$

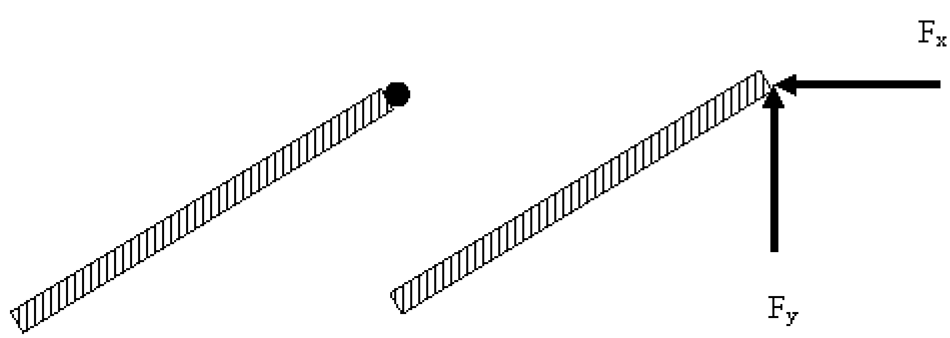
and $\sin\theta = \dfrac{(r-h)}{r}$

Solving these equations gives

$f = 0$ and $N = \dfrac{Wr}{(r-h)}$

So in this situation, we do not require friction to keep the roller in equilibrium. On the other hand recall the problem in the previous lecture when we were trying to lift a 1000Nt weights by putting a rod on a brick edge. In that case we did require friction.

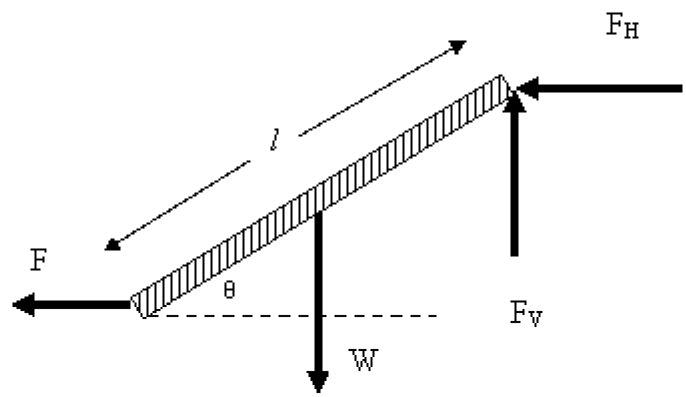
Next we consider a hinge about when an object can rotate freely. A hinge can apply a force in any direction. Thus it can apply (figure 8a) any force in X-direction and any amount of force in Y-direction but no couple.



A hinge joint and the forces applied by it

Figure 8a

To see an example, imagine lifting a train berth by pulling it horizontally. We wish to know at what angle θ from the horizontal will the berth come to equilibrium if we pull it out by a horizontal force *F* and what are the forces apply by the hinges (figure 8b).



A train berth of weight W being pulled by a horizontal force F

Figure 8b

Let the weight of the berth be *W* and its width *l*. Let the forces applied by the hinges be F_H in the horizontal direction and F_V in the vertical direction. By equilibrium conditions

$$\sum \vec{F} = 0$$

$$\sum F_X = 0 \Rightarrow -F_H - F = 0$$
$$\text{or} \quad F_H = -F$$

where the negative sign for F_H implies that it is in the direction opposite to that assumed.

Similarly

$$\sum F_y = 0 \Rightarrow F_V - W = 0 \;\; \text{or} \;\; F_V = W$$

To find the angle we apply the moment or torque balance equation about the hinges. Weight W gives a counter clockwise torque of $W\frac{l}{2}\cos\theta$ and the force F gives a clockwise torque of *Fl* sin?

$$\sum \tau = 0 \Rightarrow W\frac{l}{2}\cos\theta - Fl\sin\ \theta = 0$$

or $$\tan\theta = \frac{W}{2F}$$

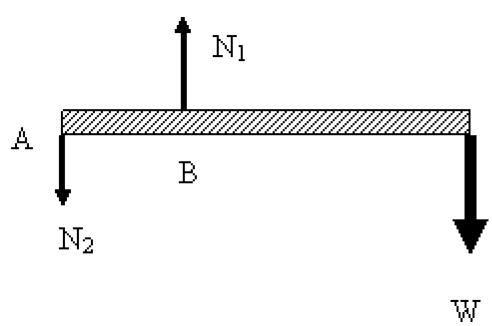
I should point put that if the hinge is not freely moving (for example due to friction) then it can produce a moment (couple) that will oppose any tendency to rotate and will have to be taken into account while considering the torque balance equation.

Next we look at a built in or fixed support as shown in the figure.

Fixed support

Figure 9

Let us analyze what happens in these cases when a load is applied. Let us look at the built-in support.



Reaction forces generated on a fixed support when a load W is applied at one end

Figure 10

As the load is put on, the beam will tend to move down on the right side pushing the inner side up. This will generate reaction forces as shown schematically in figure 10. The generated forces can be replaced by a couple and a net force either about point A or B as follows (see figure 11). Add zero force N_1 -N_1 at point A then the original N_1 and -N_1 give a couple and no force and there is a net force (N_1 -N_2) at A.

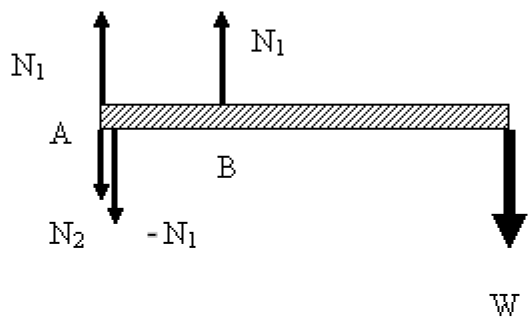
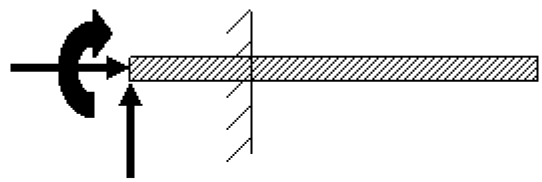


Figure 11

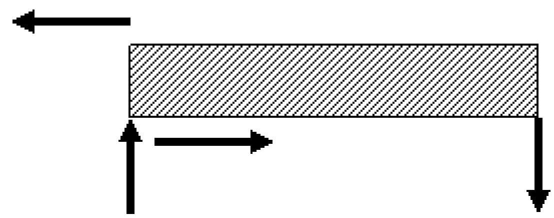
We could instead have added a zero force N_2 - N_2 at B and then would have obtained an equivalent system with a different couple moment than the previous case and a net force (N_1 - N_2) at B. I leave this for you to see. You may be wondering by now at which exactly does the force really act and what is the value of the couple. Actually in the present case the two unequal forces act on the beam so the torque provided by them is not independent of the point about which it is take. In such cases, as we will learn in the later lectures, the force effectively acts at the centroid of the force and the couple moment is equal to the torque evaluated about the centroid. In any case we can say that a built-in support provides a couple and a force. We give the schematic picture above only to motivate how the forces and the couple are generated. In reality the forces are going to be distributed over the entire portion of the support that is inside the wall and it is this distribution of force that provides a net force at the centroid and a couple equal to the torque calculated about the centroid, as we will see in later lectures. Note that deeper the support is fixed into a wall, larger would be the couple provided by it. Hence whereas to hang a light photo-frame or a painting on a wall a small nail would suffice, a longer nail would be better if the frame is heavy. In addition to providing a force perpendicular to the support and a couple, a fixed support also provides a force in the direction parallel to itself. Thus if you try to pull out the support or try to push it in, it does not move easily. The forces and couple provided by a fixed support are therefore as shown in figure 12.



Forces and couple provided by a fixed support

Figure 12

Let us now look at the support welded/ glued to the well. In that case suppose we put a load *W* at the end of the beam, you will see that the forces generated will be as shown below in figure 13.



Reaction forces generated at a glued support

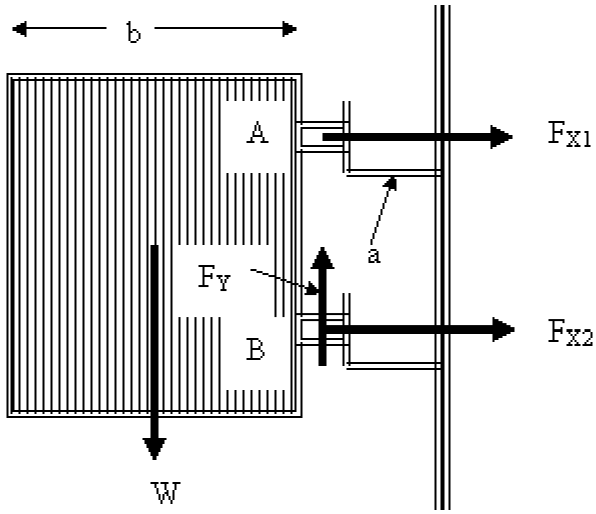
Figure 13

where in this particular case the horizontal forces must be equal so as to satisfy

$$\sum F_x = 0$$

Thus the horizontal forces provide a couple and the beam can be said to provide a couple a force in the direction perpendicular to the support. Further a glued support also cannot be pulled out or pushed in. Therefore it too is capable of providing a horizontal nonzero reaction force. Thus a welded or glued support can also be represented as shown in figure 12. Note that wider the support, larger moment it is capable of providing. Let us now solve an example of this.

Example: You must have seen gates being supported on two supports (see figure 14). Suppose the weight of the gate is *W* and its width *b*. The supports are protruding out of the wall by *a* and the distance between them is *h*. If the weight of the gate is supported fully by the lower support, find the horizontal forces, vertical forces and the moment load on both the supports.



A gate supported on two fixed supports

Figure 14

To solve this problem, let us first find out what are the forces required to keep the gate in balance. The forces applied by the supports on the gate are shown in figure 14. Since the weight of the gate is fully supported on the lower support all the vertical force is going to be provided by the lower support only. Thus

$$\sum F_y = 0 \Rightarrow F_Y - W = 0 \text{ which means } F_Y = W$$

Similarly

$$\sum F_x = 0 \Rightarrow F_{X1} + F_{X2} = 0 \text{ or } F_{X2} = -F_{X1}$$

To find, F_{X1} or F_{X2}, let us take moment about point A or B

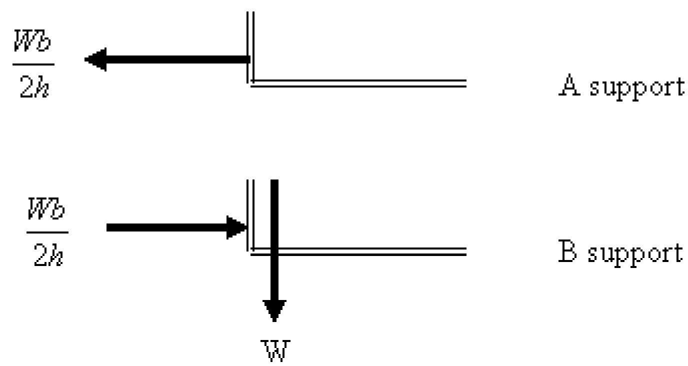
Let us make $\sum \tau_B = 0$. This gives (following the convention that counterclockwise torque is positive and clockwise torque is negative)

$$-F_{X1}h + W\frac{b}{2} = 0$$

$$\text{or } F_{X1} = \left(\frac{Wb}{2h}\right)$$

$$\text{and } F_{X2} = -\left(\frac{Wb}{2h}\right)$$

The negative sign for F_{X2} means that the force's direction is opposite to what it was taken to be in figure 14. We also wish to find the forces and couple on the support. By Newton 's IIIrd Law, forces on the support are opposite to those on the gate. Thus the forces on the two supports are:



Forces on the two supports

Figure 15

You see that support A is being pulled out whereas support B is being pushed in (we observe an effect of this at our houses all the time: the upper hinges holding a door tend to come out of the doorjamb). Now the force by the wall on support A will be $\left(\frac{Wb}{2h}\right)$- to the right to keep it fixed in its place. On the other hand the situation for the lower support is more involved. The lower support will be kept in its place by the wall providing it horizontal and vertical forces and a torque. The net horizontal force is $\left(\frac{Wb}{2h}\right)$- to the left and the net vertical force is W pointing up. The lower support also balances a torque. Taking the torques about the point where it enters the wall, its value comes out to be

$$\tau = Wa$$

If we assume that the net vertical force and the torque is provided by only two reaction forces at two points as in figure 10, these two reaction forces can be calculated easily if we know the length of the portion inside the wall. I leave it as an exercise. In solving this, you will notice that the reaction forces are smaller if the support is deeper inside the wall. As pointed out earlier, in reality the force is going to be distributed over the entire portion of the support inside the wall. So a more realistic calculation is a little more involved.

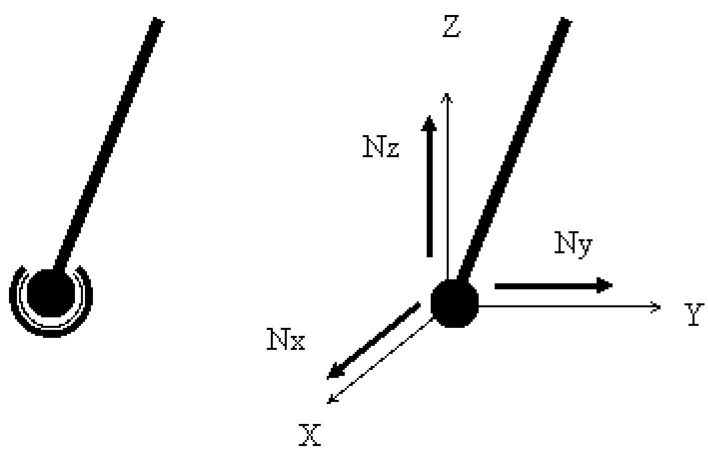
To summarize this lecture, we have looked at some simple engineering elements and have outlined what kind of forces and torques are they capable of applying. In the next lecture we are going discuss forces in three dimensions. We are also going to look at conditions that forces with certain geometric relations should satisfy for providing equilibrium.

Lecture 5
Equilibrium of bodies III

In the previous lecture I have been talking about equilibrium in a plane. We now move on to three dimensional (*3-d*) cases. In three dimensional cases the equilibrium conditions lead to balance along all three axes. Then

$$\sum \vec{F} = 0 \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases} \quad \text{and} \quad \sum \vec{\tau} = 0 \Rightarrow \begin{cases} \sum \tau_x = 0 \\ \sum \tau_y = 0 \\ \sum \tau_z = 0 \end{cases}$$

We now have to take care of components of forces and torque in all three dimensions. The engineering elements that we considered earlier are now considered as *3-d* case. Thus consider a ball-socket joint in which a ball is supported in a socket (figure 1).

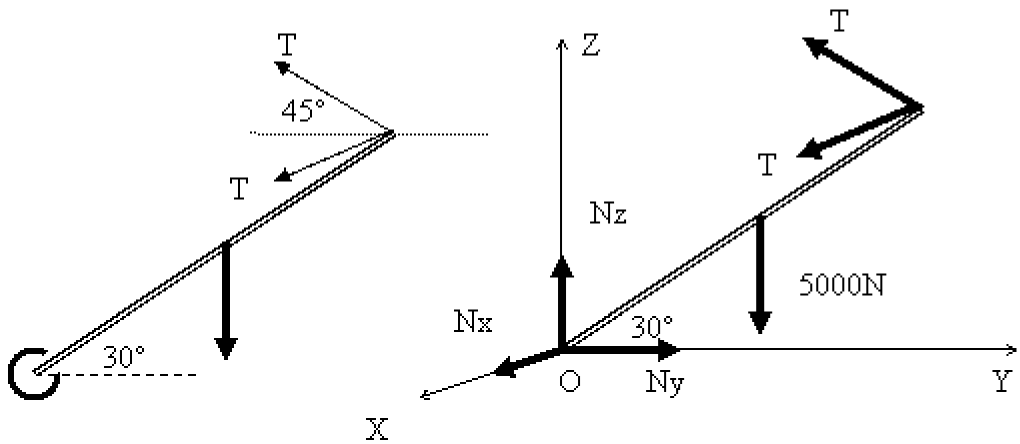


A ball-socket joint (left) is capable of applying forces in all three directions (right).

Figure 1

A ball-socket joint provides reaction forces *Nx*, *Ny* and *Nz* in all three directions (figure 1) but it cannot apply any torque. This is a little like a hinge joint in *2d* . Let me solve an example using such a joint.

Example 1: To balance a heavy weight of 5000 *N*, two persons dig a hole in the ground and put a pole of length *l* in it so that the hole acts as a socket. The pole makes an angle of 30° from the ground. The weight is tied at the mid point of the pole and the pole is pulled by two horizontal ropes tied at its ends as shown in figure 2. Find the tension in the two ropes and the reaction forces of the ground on the pole.



A pole balancing a weight on it (left). Forces acting on it are shown on the right.

Figure 2

To solve this problem, let me first choose a co-ordinate system. I choose it so that the pole is over the y-axis in the *(y-z)* plane (see figure 2).

The ropes are in *(xy)* direction with tension *T* in each one of them so that tension in each is written as

$$\left(\frac{T}{\sqrt{2}}\hat{i}-\frac{T}{\sqrt{2}}\hat{j}\right) \quad \text{and} \quad \left(-\frac{T}{\sqrt{2}}\hat{i}-\frac{T}{\sqrt{2}}\hat{j}\right)$$

You may be wondering why I have taken the tension to be the same in the two ropes. Actually it arises from the torque balance equation; if the tensions were not equal, their component in the x-direction will give a nonzero torque.

Let the normal reaction of the ground be (*Nx , Ny , Nz*). Then the force balance equation gives

$$\sum F_x = 0 \Rightarrow N_x + \frac{T}{\sqrt{2}} - \frac{T}{\sqrt{2}} = 0 \Rightarrow N_x = 0$$

$$\sum F_y = 0 \Rightarrow N_y - \frac{2T}{\sqrt{2}} = 0 \Rightarrow N_y = T\sqrt{2}$$

$$\sum F_z = 0 \Rightarrow N_z - 5000 = 0 \Rightarrow N_z = 5000N$$

Taking torque about point O and equating it to zero, we get

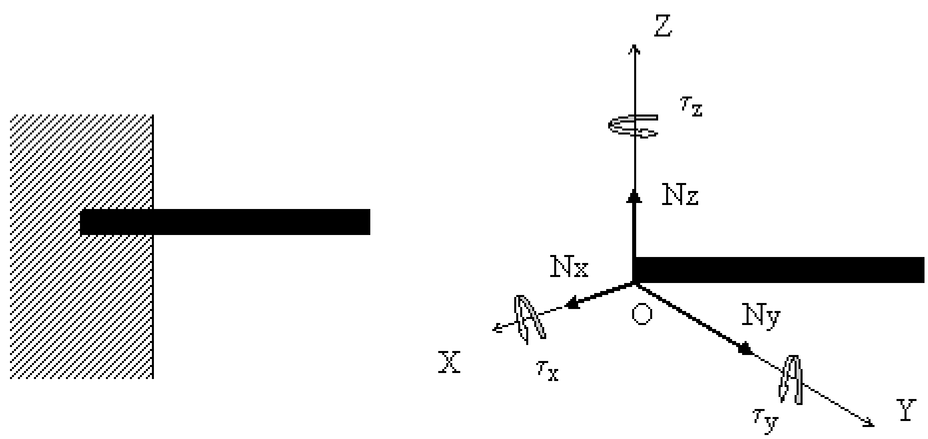
$$\left(l\cos 30^o\,\hat{j} + l\sin 30^o\,\hat{k}\right)\times\left(-T\sqrt{2}\,\hat{j}\right) + \left(\frac{l}{2}\cos 30^o\,\hat{j} + \frac{l}{2}\sin 30^o\,\hat{k}\right)\times\left(-5000\hat{k}\right) = 0$$

$$Tl\sqrt{2}\,\frac{1}{2}\hat{i} - 2500\cdot l\cdot\frac{\sqrt{3}}{2}\hat{i} = 0 \quad\Rightarrow\quad T\sqrt{2} = 2500\sqrt{3}$$

which gives

$$\boxed{T = 3062N. \qquad N_y = 4330N}$$

Next, if I consider a fixed connection, say in a wall, it is capable of providing force along all the three axes and also of providing torques about the three axes, Thus in *3-d* it will be represented as shown in figure 3.



A fixed joint (left) is capable of producing reaction forces and reaction torques along all three axes (right).

Figure 3

This is a generalization of the fixed or welded/glued support in *2-d* . How are these torques etc. generated? Recall what I did for a fixed support in *2-d* and carry out a similar analysis in *3-d.*

Hopefully by the analysis carried out so far, you would be able to recognize what all reactions a given element of a mechanical system can provide. For example look at the support shown in figure 4 where the shaft can not move through the hole in the fixed block, but it is free to rotate. Can you tell the reaction forces and torques that this support provides?

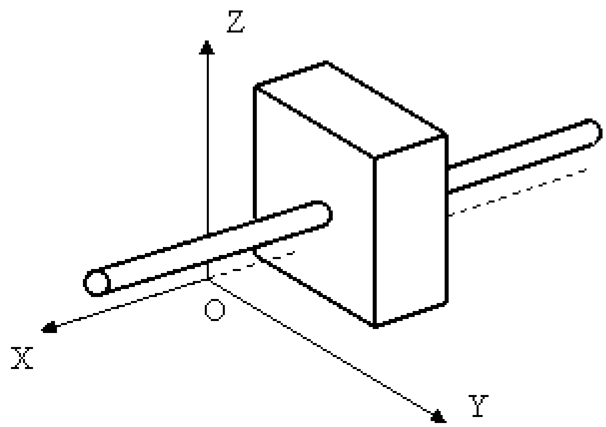


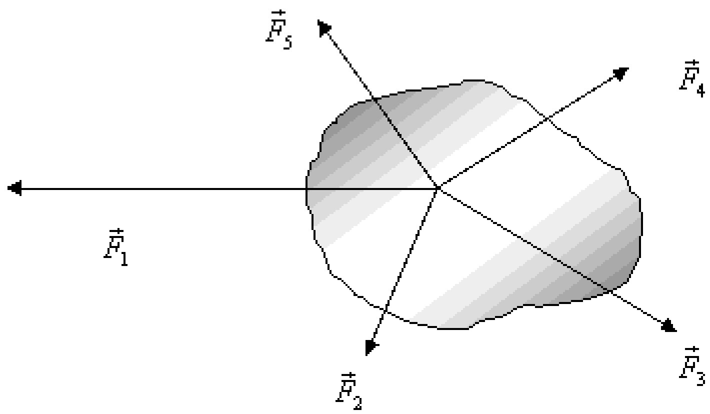
Figure 4

Having discussed the elements that apply different kinds of forces, let us look at some situations there due to the geometry of forces applied, some of the equilibrium equations are automatically satisfied. If we recognize this, it saves us from doing extra calculations involving that particular condition.

If all forces are **concurrent at a point** (see figure 5), i.e., they all cross each other at one point *O* then torques of all the forces is identically zero about *O* . Thus the only equilibrium condition is

$$\sum \vec{F} = 0 \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases}$$

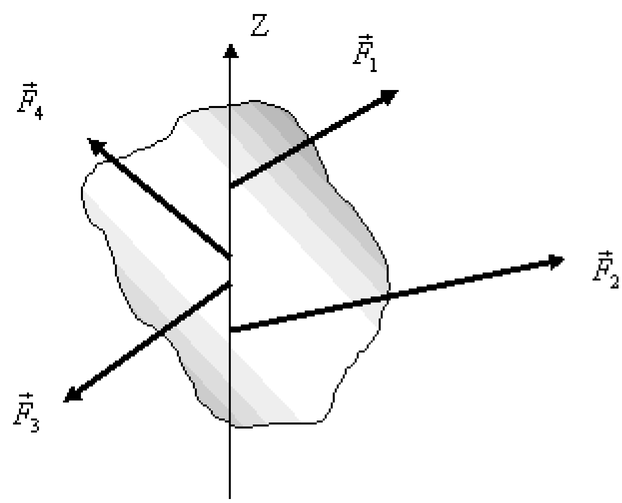
Recall that if the sum of all forces on a system is zero, torque is independent of the origin. Thus although in the beginning I used the fact that torque about the point of concurrence is zero, it is true about any point once the force equation for equilibrium is satisfied.



Forces concurrent at a point

Figure 5

Next consider the case when all **forces intersect one particular line**, call it the z-axis without any loss of generality (see figure 6).



Forces intersecting one line

Figure 6

Using transmissibility of the force, in this case we can take the force $\vec{F}_i \ (i = 1, 2, \cdots)$ to be acting at point $Z_i\hat{k}$. Then the torque due to all these forces will be

$$
\begin{aligned}
\vec{\tau} &= \sum_i Z_i\hat{k}\times\left(F_{ix}\hat{i}+F_{iy}\hat{j}+F_{iz}\hat{k}\right)\\
&= \sum_i \left(Z_iF_{ix}\hat{j}-Z_iF_{iy}\hat{i}\right)
\end{aligned}
$$

Thus the Z component of the torque is automatically zero. In general when the forces intersect a line, the torque component along that line vanishes. Under these circumstances, if we take that line to be the z-axis, the equilibrium conditions are

$$
\sum\vec{F}=0\Rightarrow\begin{cases}\sum F_x=0\\ \sum F_y=0\\ \sum F_z=0\end{cases}\qquad\text{and}\qquad\sum\vec{\tau}=0\Rightarrow\begin{cases}\sum\tau_x=0\\ \sum\tau_y=0\end{cases}
$$

Next I discuss what happens if all the applied forces are parallel, say to the Z axis. Then the forces do not have any *x or y* components. Further, by the z-component of the torque also vanishes (left as an exercise for you to show). The equilibrium conditions in this case reduce to

$$
\sum\vec{F}=0\Rightarrow\sum F_z=0\qquad\text{and}\qquad\sum\vec{\tau}=0\Rightarrow\begin{cases}\sum\tau_x=0\\ \sum\tau_y=0\end{cases}
$$

In general of course we have all the six condition.

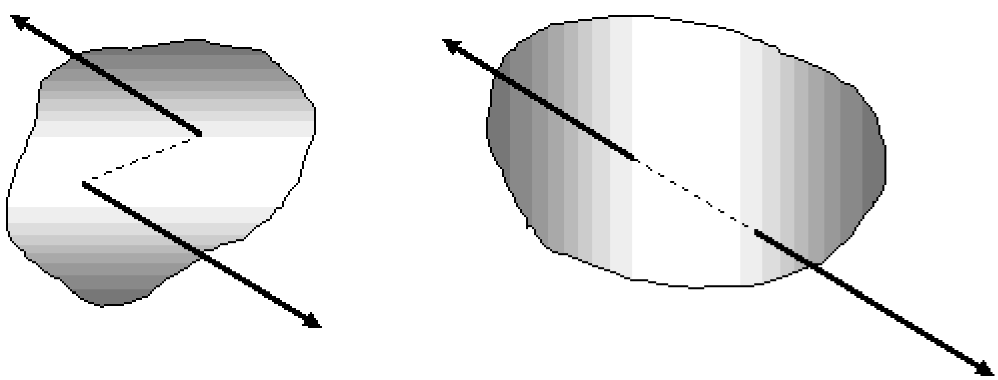
$$
\sum\vec{F}=0\Rightarrow\begin{cases}\sum F_x=0\\ \sum F_y=0\\ \sum F_z=0\end{cases}\qquad\text{and}\qquad\sum\vec{\tau}=0\Rightarrow\begin{cases}\sum\tau_x=0\\ \sum\tau_y=0\\ \sum\tau_z=0\end{cases}
$$

Let me now summarize what all you have learnt so far in considering the equilibrium of engineering structures. In the process I also introduce you to a term called the **Free Body Diagram**. I have actually been using it without calling it so. Now, let us formalize it.

In talking about the equilibrium of a body we consider all the external forces applied on it and the interaction of the body with other objects around it. This interaction produces more forces and torques on the body. Thus when we single out a body in equilibrium, objects like hinges, ball-socket joint, fixed supports around it are replaced elements by the corresponding forces & torques that they generate. This is what is called a free-body diagram. Making a free-body diagram allows us to focus our attention only on the information relevant to the equilibrium of the body, leaving out unnecessary details. Thus making a free-body diagram is pretty much like Arjuna - when asked to take an aim on the eye of a bird - seeing only the eye and nothing else. The diagrams made on the right side of figures 1, 2 and 3 are all free-body diagrams.

In the coming lecture we will be applying the techniques learnt so far to a very special structure called the truss. To prepare you for that, in the following I consider the special case of a system in equilibrium under only two forces. For completeness I will also take up equilibrium under three forces.

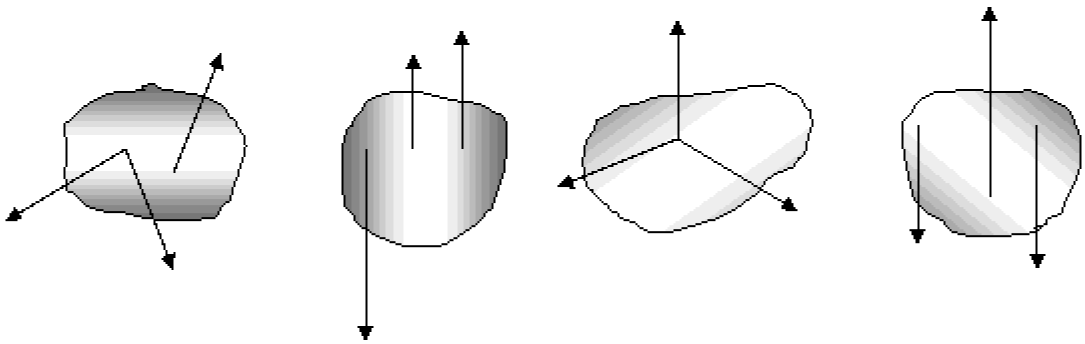
When only two forces are applied, no matter what the shape or the size of the object in equilibrium is, the forces must act along the same line, in directions opposite to each other, and their magnitudes must be the same. That the forces act in directions opposite to each other and have equal magnitude follows from the equilibrium conditions $\vec{F}_1 + \vec{F}_2 = 0$, which implies that $\vec{F}_2 = -\vec{F}_1$. Further, if the forces are not along the same line then they will form a couple that will tend to rotate the body. Thus $\vec{\tau} = 0$ implies that the forces act along the same line, i.e. they be collinear (see figure 7).



Two bodies being applied two equal and opposite forces. The body on the left is not in equilibrium whereas that on the right is.

Figure 7

Similarly if there are three forces acting on a body that is in equilibrium then the three forces must be in the same plane and concurrent. If there are not concurrent then they must be parallel (of course remaining in the same plane). This can be understood as follows. Any two members of the three applied forces form a plane. If the third force is not in the same plane, it will have a non-vanishing component perpendicular to the plane; and that component does not get cancelled. Thus unless all three forces are in the same plane, they cannot add up to zero. So to satisfy the equation $\sum \vec{F} = 0$, the forces must be in the same plane, i.e. they must be coplanar. For equilibrium the torque about any point must also be zero. Since the forces are in the same plane, any two of them will intersect at a certain point *O*. These two forces will also have zero moment about *O*. If the third force does not pass through *O*, it will give a non-vanishing torque (see figure 8). So to satisfy the torque equation, the forces have to be concurrent. Zero torque condition can also be satisfied if the three forces are parallel forces (see figure 8); that is the other possibility for equilibrium under three forces.

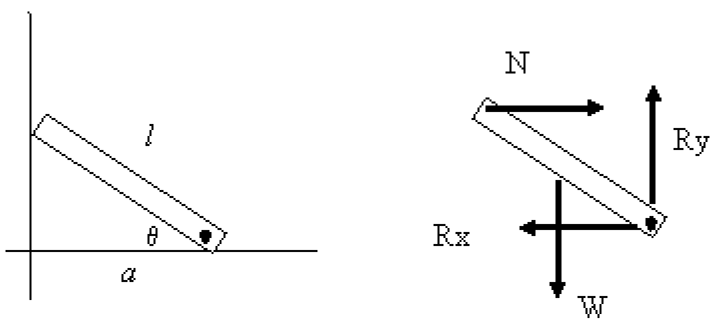


Four bodies being applied three coplanar forces. Two bodies on the left are not in equilibrium whereas the two on the right are.

Figure 8

In the end, I now discuss one more concept about equilibrium of bodies, that of **statical determinacy** . Along the way I also introduce some connected concepts like constraints, degree of redundancy and redundant support. On constraints, I will discuss more in the lecture on Method of virtual work.

To introduce the terms used above, I consider a rod of length *l* and weight *W* held at a pin-joint on a floor at a distance of *a* from a wall, on which its other end is. This is shown in figure 9 along with the free-body diagram of the rod.



A rod of length l and weight W held at a pin-joint (left). Its free body diagram is drawn on the right.

Figure 9

There are three unknowns - Rx, Ry and N - in the problem and three equations of equilibrium that will determine the unknowns. Specifically:

$$R_y - W = 0 \Rightarrow R_y = W$$

$$R_x - N = 0 \Rightarrow R_x = N$$

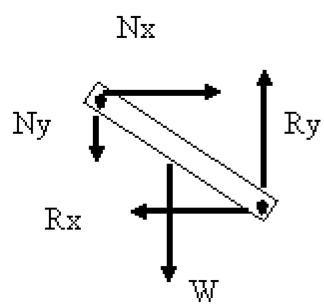
Taking moment about the pin, we get

$$N\sqrt{l^2 - a^2} = W \cdot \frac{a}{2} \quad \Rightarrow \quad N = \frac{Wa}{2\sqrt{l^2 - a^2}}$$

This gives

$$\boxed{R_x = N = \frac{Wa}{2\sqrt{l^2 - a^2}} \quad \text{and} \quad R_y = W}$$

In this case, the constraints or the external supports we apply are just sufficient to keep the system in equilibrium. Such systems are known as **statically determinate** systems. Now suppose we apply one more support. Let us support the rod at both ends by pin joints. The free-body diagram will then look like that shown in figure 10.



Free body diagram of the rod when it is supported by pin joints at both its ends.

Figure 10

Now the pin on top end is also applying a force on the rod. Thus the equations of equilibrium read as

$$-R_x + N_x = 0 \qquad R_y - N_y = W$$

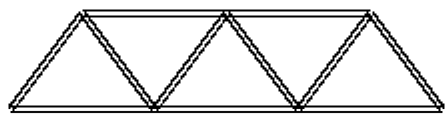
$$-N_x\sqrt{l^2 - a^2} + N_y a + \frac{Wa}{2} = 0$$

The situation on hand is that we have four unknowns - Rx, Ry, Nx and Ny - and only three equations. Thus one of the unknown cannot be determined. In particular only $R_y - N_y = W$ is known and what are individual R_y and N_y cannot be determined unless some additional information is also given. Such systems are known as **statically indeterminate** systems. In such systems we are applying more constraints than are needed to keep the system in equilibrium. Even if we remove one of the constraints - in this case replace the upper pin by a plane surface - the system is capable of remaining in equilibrium. Such supports that can be removed without disturbing the equilibrium are known as redundant supports. And the number of redundant supports is the **degree of statical indeterminacy** .

After introducing you to the concepts discussed above, we will be studying trusses in the next lecture.

Lecture 6
Trusses

Having set up the basics for studying equilibrium of bodies, we are now ready to discuss the trusses that are used in making stable load-bearing structures. The examples of these are the sides of the bridges or tall TV towers or towers that carry electricity wires. Schematic diagram of a structure on the side of a bridge is drawn in figure 1.

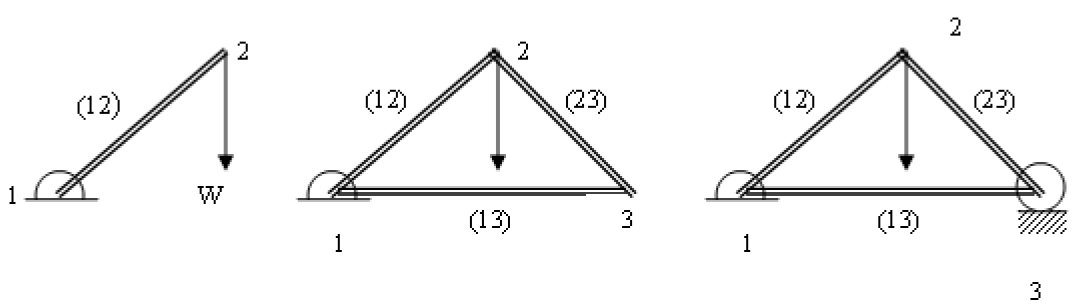


Side of a bridge

Figure 1

The structure shown in figure 1 is essentially a two-dimensional structure. This is known as a plane truss. On the other hand, a microwave or mobile phone tower is a three-dimensional structure. Thus there are two categories of trusses - Plane trusses like on the sides of a bridge and space trusses like the TV towers. In this course, we will be concentrating on plane trusses in which the basis elements are stuck together in a plane.

To motivate the structure of a plane truss, let me take a slender rod (12) between points 1 and 2 and attach it to a fixed pin joint at 1 (see figure 2).



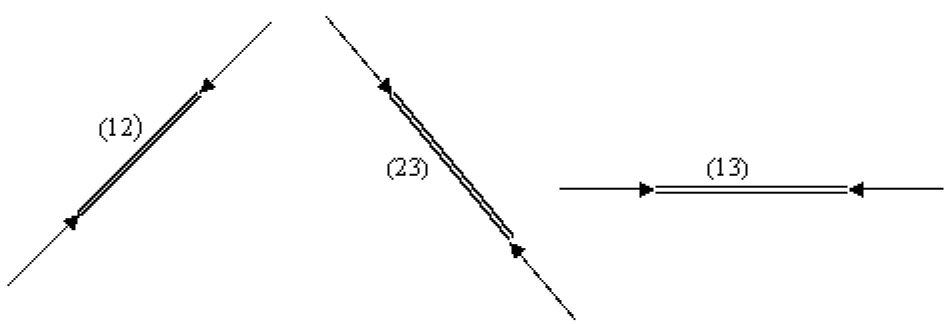
Developing a plane truss (left to right)

Figure 2

Now I put a pin (pin2) at point 2 at the upper end and hang a weight W on it. The question is if we want to hold the weight at that point, what other minimum supports should we provide? For rods we are to make only pin joints (We assume everything is in this plane and the structures does not topple side ways). Since rod (12) tends to turn clockwise, we stop the rightward movement of point 2 by connecting a rod (23) on it and then stop point 3 from moving to the right by connecting it to point 1 by another rod (13). All the joints in this structure are pin joints. However, despite all this the entire structure still has a tendency to turn to turn clockwise because there is a torque on it due to W. To counter this, we attach a wheel on point 3 and put it on the ground. This is the bare minimum that we require to hold the weight is place. The triangle made by rods forms the basis of a plane truss.

Note: One may ask at this point as to why as we need the horizontal rod (13). It is because point 3 will otherwise keep moving to the right making the whole structure unstable. Rod (13) has two forces acting on it: one vertical force due to the wheel and the other at end 2. However these two forces cannot be collinear so without the rod (13) the system will not be in equilibrium. Generally, in a truss each joint must be connected to at least three rods or two rods and one external support.

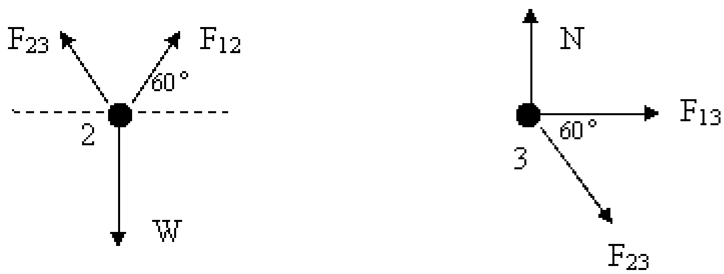
Let us now analyze forces in the structure that just formed. For simplicity I take the lengths of all rods to be equal. To get the forces I look at all the forces on each pin and find conditions under which the pins are in equilibrium. The first thing we note that each rod in equilibrium under the influence of two forces applied by the pins at their ends. As I discussed in the previous lecture, in this situation the forces have to be collinear and therefore along the rods only. Thus each rod is under a tensile or compressive force. Thus rods (12), (23) and (13) experience forces as shown in figure 3.



Forces on the three rods of triangle formed in figure 2

Figure 3

Notice that we have taken all the forces to be compressive. If the actual forces are tensile, the answer will come out to be negative. Let us now look at pin 2. The only forces acting on pin 2 are F_{12} due to rod (12) and F_{23} due to rod (23). Further, it is pulled down by the weight W. Thus forces acting on pin 2 look like shown in figure 4.



Forces on pin 2 and pin 3

Figure 4

Applying equilibrium condition to pin (2) gives

$$F_{12}\cos 60^o = F_{23}\cos 60^o \quad \text{or} \quad F_{12} = F_{23}$$

$$2F_{12}\sin\ 60^o = W \quad \Rightarrow \quad F_{12} = \frac{W}{\sqrt{3}} = F_{23}$$

Let us now look at pin 3 (see figure 4). It is in equilibrium under forces F_{23}, normal reaction N and a horizontal force F_{13}.

Applying equilibrium condition $\left(\sum \vec{F} = 0\right)$ gives

$$F_{23}\cos 60^o + F_{13} = 0$$

$$\Rightarrow F_{12} = -F_{23}\cos 60^o$$

$$= -\frac{W}{2\sqrt{3}}$$

Since the direction of F_{13} is coming out to be negative, the direction should be opposite to that assumed. Balance of forces in the vertical direction gives

$$N - F_{23}\sin\ 60^o = 0 \Rightarrow N - \frac{W}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 0 \quad \text{or} \quad N = \frac{W}{2}$$

Thus we see that the weight is held with these three rods. The structure is determinate and it holds the weight in place.

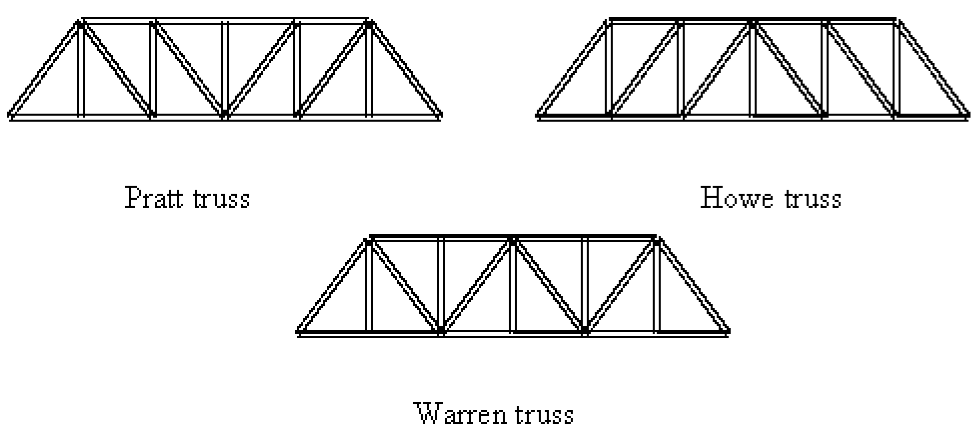
Even if we replace the pin joints by a small plate (known as gusset plate) with two or three pins in these, the analysis remains pretty much the same because the pins are so close together that they hardly create any moment about the joints. Even if the rods are welded together at the joints, to a great degree of accuracy most of the force is carried longitudinally on the rods, although some very small (negligible) moment is created by the joints and may be by possible bending of the rods.

Now we are ready to build a truss and analyze it. We are going to build it by adding more and more of triangles together. As you can see, when we add these triangles, the member of joints *j* and the number of members (rods) *m* are related as follows:

$m = 2j - 3$

This makes a truss statically determinate. This is easily understood as follows. First consider the entire truss as one system. If it is to be statically determinate, there should be only three unknown forces on it because for forces in a plane there are three equilibrium conditions. Fixing one of its ends a pin joint and putting the other one on a roller does that (roller also gives the additional advantage that it can help in adjusting any change in the length of a member due to deformations). If we wish to determine these external forces and the force in each member of the truss, the total number of unknowns becomes $m + 3$. We solve for these unknowns by writing equilibrium conditions for each pin; there will be $2j$ such equations. For the system to be determinate we should have $m + 3 = 2j$, which is the condition given above. If we add any more members, these are redundant. On the other hand, less number of members will make the truss unstable and it will collapse when loaded. This will happen because the truss will not be able to provide the required number of forces for all equilibrium conditions to be satisfied. Statically determinate trusses are known as simple trusses.

Exercise 1: Shown in figure 5 are three commonly used trusses on the sides of bridges. Show that all three of them are simple trusses.



Three commonly used trusses on the sides of bridges

Figure 5

You may ask why we put trusses on bridges. As our later analysis will show they distribute the load over all elements and thereby making the bridge stronger.

We now wish to obtain the forces generated in various arms of a truss when it is loaded externally. This is done under the following assumptions:

1. If the middle line of the members of a truss meet at a point that point is taken as a pin joint. This is a very god assumption because as we have seen earlier while introducing a truss (triangle with pin joint), the load is transferred on to other member of the trusses so that forces remain essentially collinear with the member.

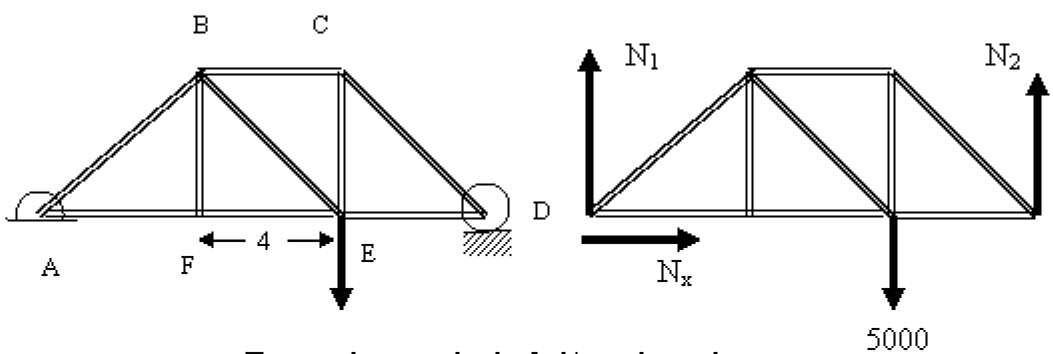
2. All external loads are applied on pin connections.

3. All members' weight is equally divided on connecting pins.

There are two methods of determining forces in the members of a truss - Method of joints and method of sections. We start with the method of joints:

Method of joints: In method of joints, we look at the equilibrium of the pin at the joints. Since the forces are concurrent at the pin, there is no moment equation and only two equations for equilibrium viz. $\sum F_x = 0$ and $\sum F_y = 0$. Therefore we start our analysis at a point where one known load and at most two unknown forces are there. The weight of each member is divided into two halves and that is supported by each pin. To an extent, we have already alluded to this method while introducing trusses. Let us illustrate it by two examples.

Example 1: As the first example, I take truss ABCDEF as shown in figure 6 and load it at point E by 5000N. The length of small members of the truss is 4m and that of the diagonal members is $4\sqrt{2}$ m. I will now find the forces in each member of this truss assuming them to be weightless.



Truss of example 1 (left) and its free body diagram (right)

Figure 6

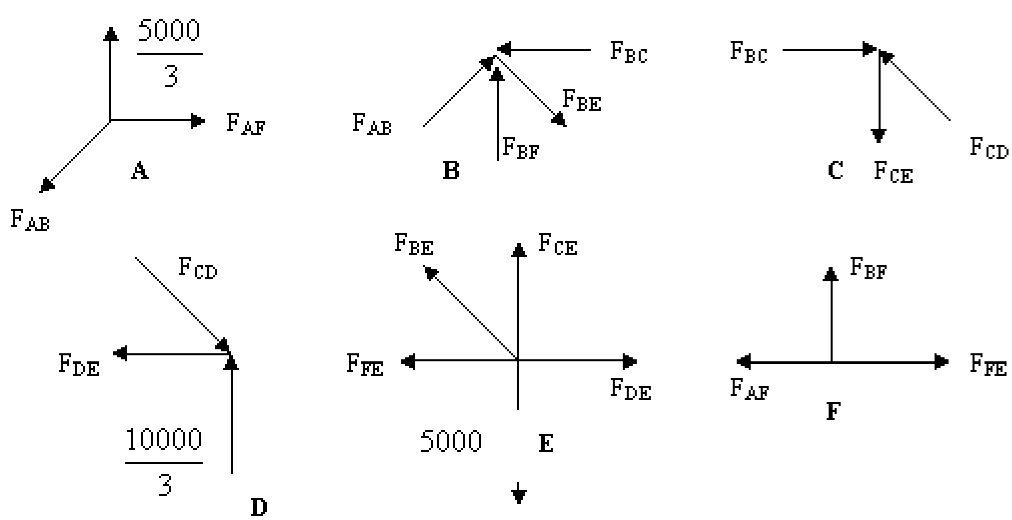
We take each point to be a pin joint and start balancing forces on each of the pins. Since pin E has an external load of 5000N one may want to start from there. However, E point has more than 2 unknown forces so we cannot start at E. We therefore first treat the truss as a whole and find reactions of ground at points A and D because then at points A and D their will remain only two unknown forces. The horizontal reaction *Nx* at point A is zero because there is no external horizontal force on the system. To find N_2 I take moment about A to get

$$N_2 = \frac{10000}{3} N$$

which through equation $\sum F_y = 0$ gives

$$N_1 = \frac{5000}{3} N$$

In method of joints, let us now start at pin A and balance the various forces. We already anticipate the direction and show their approximately at A (figure 7). All the angles that the diagonals make are 45° .



Forces at various joints of the truss in figure 6

Figure 7

The only equations we now have worry about are the force balance equations.

$$\sum F_y = 0 \quad gives \quad \frac{F_{AB}}{\sqrt{2}} = \frac{5000}{3} \quad or \quad F_{AB} = 2355N$$

$$now \sum F_x = 0 \quad gives \quad F_{AF} = \frac{F_{AB}}{\sqrt{2}} = \frac{5000}{3} N$$

Keep in mind that the force on the member AB and AF going to be opposite to the forces on the pin (Newton 's IIIrd law). Therefore force on member AB is compressive (pushes pin A away) whereas that on AF is tensile (pulls A towards itself).

Next I consider joint F where force AF is known and two forces BF and FE are unknown. For pint F

$$\sum F_x = 0 \quad gives \quad F_{FE} = F_{AF} = \frac{5000}{3}(tensile)$$
$$\sum F_y = 0 \quad gives \quad F_{BF} = 0 (No\,Force\,on\,BF)$$

Next I go to point B since now there are only two unknown forces there. At point B

$$\sum F_y = 0 \quad gives \quad F_{AB}\cos 45^o + F_{BE}\cos 45^o = 0$$
$$\text{or} \quad F_{BE} = -F_{AB} = -2355N$$

Negative sign shows that whereas we have shown F_{BE} to be compressive, it is actually tensile.

$$\sum F_x = 0 \quad \Rightarrow \quad F_{BC} - F_{AB}\sin 45^o - F_{BE}\sin 45^o = 0$$
$$F_{BC} = \frac{F_{AB}}{\sqrt{2}} + \frac{F_{BE}}{\sqrt{2}} = \frac{10{,}000}{3}N.(compressive)$$

Next I consider point C and balance the forces there. I have already anticipated the direction of the forces and shown F_{CE} to be tensile whereas F_{CD} to be compressive

$$\sum F_x = 0 \quad gives \quad F_{BC} = \frac{F_{CD}}{\sqrt{2}} \Rightarrow F_{CD} = F_{BC}\sqrt{2} = 4710N$$
$$\sum F_Y = 0 \quad gives \quad F_{CE} = \frac{F_{CD}}{\sqrt{2}} = \frac{F_{BC}\sqrt{2}}{\sqrt{2}} = F_{BC} = \frac{10{,}000}{3}N$$

Next I go to pin D where the normal reaction is $\frac{10{,}000}{3}$ N and balance forces there.

$$\sum F_Y = 0 \quad gives \quad \frac{F_{CD}}{\sqrt{2}} = \frac{10{,}000}{3}N$$
$$\sum F_x = 0 \quad gives \quad F_{DE} = \frac{F_{CD}}{\sqrt{2}} = \frac{10{,}000}{3}$$

Thus forces in various members of the truss have been determined. They are

$$F_{AB}=\frac{5000\sqrt{2}}{3}N(compressive), F_{AF}=\frac{5000}{3}N(Tensile), F_{BF}=0$$

$$F_{FE}=\frac{5000}{3}N(Tensile), F_{BC}=\frac{10,000}{3}N(Compressive), F_{BE}=\frac{5000\sqrt{2}}{3}N(Tensile)$$

$$F_{CE}=\frac{10,000}{3}N(Tensile), F_{CD}=\frac{10,000\sqrt{2}}{3}N\ (compressive), F_{DE}=\frac{10,000}{3}N(Tensile)$$

You may be wondering how we got all the forces without using equations at all joints. Recall that is how we had obtained the statical determinacy condition. We did not have to use all joints because already we had treated the system as a whole and had gotten two equations from there. So one joint - in this case E - does not have to be analyzed. However, given that the truss is statically determinate, all these forces must balance at point E, where the load has been applied, also. I will leave this as an exercise for you. Next I ask how the situation would change if each member of the truss had weight. Suppose each members weighs 500N, then assuming that the load is divided equally between two pins holding the member the loading of the truss would appear as given in figure 8 (loading due to the weight as shown in red). Except at points A and D the loading due to the weight is 750N; at the A and D points it is 500N.

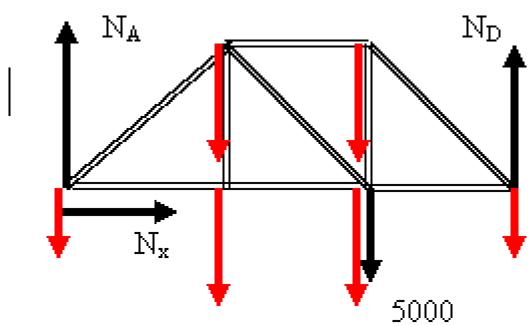


Figure 8

Now the external reaction at each end will be.

$$N_A=\frac{5000}{3}+2000=\frac{11,000}{3}N$$

$$N_D=\frac{10,000}{3}+2000=\frac{16,000}{3}N$$

The extra 2000N can be calculated either from the moment equation or straightaway by realizing that the new added weight is perfectly symmetric about the centre of the truss and therefore will be equally divided between the two supports. For balancing forces at other pins, we follow the same procedure as above, keeping in mind though that each pin now has an external loading due to the weight of each member. I'll solve for forces in some member of the truss. Looking at pin A, we get

$$\sum F_Y = 0 \quad \Rightarrow \quad \frac{F_{AB}}{\sqrt{2}} = \frac{11{,}000}{3} - 500$$

$$\text{or} \quad F_{AB} = \frac{9500\sqrt{2}}{3}(compressive)$$

$$F_{AF} = \frac{F_{AB}}{\sqrt{2}} = \frac{9500}{3} N(tensile).$$

Next we move to point F and see that the forces are

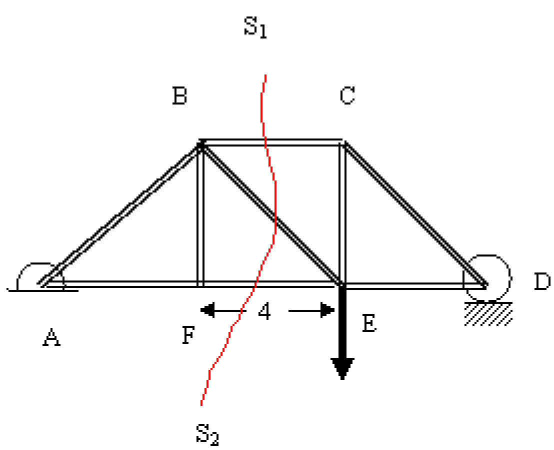
$$\sum F_x = 0 \quad \Rightarrow \quad F_{FE} = F_{AF} = \frac{10{,}000}{3} N(tensile)$$

$$\sum F_Y = 0 \quad \Rightarrow \quad F_{FB} = 750 N(tensile)$$

One can similarly solve for other pins in the truss and I leave that as an exercise for you.

Having demonstrated to you the method of joints, we now move on to see the method of sections that directly gives the force on a desired member of the truss.

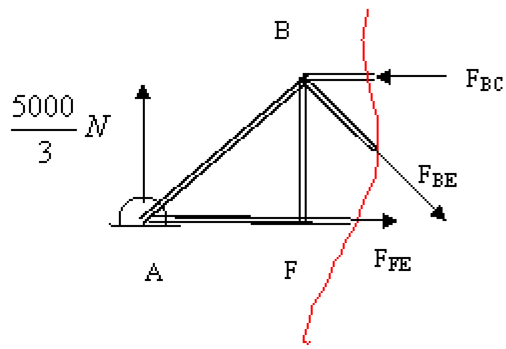
Method of sections : As the name suggests in method of sections we make sections through a truss and then calculate the force in the members of the truss though which the cut is made. For example, if I take the problem we just solved in the method of joints and make a section S_1, S_2 (see figure 9), we will be able to determine the forces in members BC, BE and FE by considering the equilibrium of the portion to the left or the right of the section.



A cut made through a truss to apply the method of sections

Figure 9

Let me now illustrate this. As in the method of joints, we start by first determining the reactions at the external support of the truss by considering it as a whole rigid body. In the present particular case, this gives $\frac{10000}{3}$ *N* at D and $\frac{5000}{3}$ *N* at A. Now let us consider the section of the truss on the left (see figure 10).



Left section of the truss taken to apply method of sections

Figure 10

Since this entire section is in equilibrium, $\sum F_x = 0, \sum F_y = 0$ and $\sum \tau = 0$. Notice that we are now using all three equations for equilibrium since the forces in individual members are not concurrent. The direction of force in each member, one can pretty much guess by inspection. Thus the force in the section of members BE must be pointing down because there is no other member that can give a downward force to counterbalance $\frac{5000}{3}$ *N* reaction at A. This clearly tells us that *F BE* is tensile. Similarly, to counter the torque about B generated by $\frac{5000}{3}$ *N* force at A, the force on FE should also be from F to E. Thus this force is also tensile. If we next consider the balance of torque about A, $\frac{5000}{3}$ N and F_{FE} do not give any torque about A. So to counter torque generated by F_{BE} , the force on BC must act towards B, thereby making the force compressive.

Let us now calculate individual forces. F_{FE} is easiest to calculate. For this we take the moment about B. This gives

$$4 \times \frac{5000}{3} = 4 \times F\ FE$$

$$F\ FE = \frac{5000}{3}\ N$$

Next we calculate F_{BE} . For this, we use the equation $\sum F_Y = 0$. It gives

$$\frac{F_{BE}}{\sqrt{2}} = \frac{5000}{3} N \quad \text{or} \quad F_{BE} = \frac{5000\sqrt{2}}{3} N$$

Finally to calculate F_{BC} , we can use either the equation $\sum \tau = 0$ about A or $\sum F_x = 0$

$$\sum F_x = 0 \quad \text{gives} \quad \frac{F_{BE}}{\sqrt{2}} + F_{FE} = F_{BC}$$

$$\Rightarrow F_{BC} = \frac{5000}{3} + \frac{5000}{3} = \frac{10{,}000}{3} N$$

Thus we have determined forces in these three members directly without calculating forces going from one joint to another joint and have saved a lot of time and effort in the process. The forces on the right section will be opposite to those on the left sections at points through which the section is cut. This can be used to check our answer, and I leave it as an exercise for you.

After this illustration let me put down the steps that are taken to solve for forces in members of a truss by method of sections:

1. Make a cut to divide the truss into section, passing the cut through members where the force is needed.

2. Make the cut through three member of a truss because with three equilibrium equations viz. $\sum F_x = 0, \sum F_Y = 0 \text{ and } \sum \tau = 0$ we can solve for a maximum of three forces.

3. Apply equilibrium conditions and solve for the desired forces.

In applying method of sections, ingenuity lies in making a proper. The method after a way of directly calculating desired force circumventing the hard work involved in applying the method of joints where one must solve for each joint.

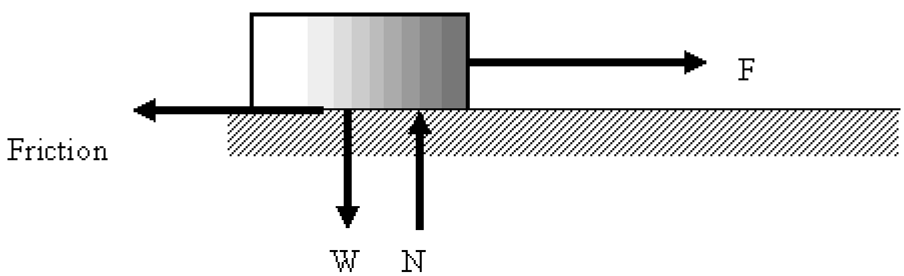
We thus conclude one lecture or trusses. Next step in making the treatment accurate is obviously to take care of deformation in the members of a truss. This will be done in an advanced course later.

Lecture 7
Friction

Whatever we have studied so far, we have always taken the force applied by one surface on an object to be normal to the surface. In doing so, we have been making an approximation i.e., we have been neglecting a very important force viz., the frictional force. In this lecture we look at the frictional force in various situations.

In this lecture when we talk about friction, we would mean frictional force between two dry surfaces. This is known as Coulomb friction. Frictional forces also exist when there is a thin film of liquid between two surfaces or within a liquid itself. This is known as the viscous force. We will not be talking about such forces and will focus our attention on Coulomb friction i.e., frictional forces between two dry surfaces only. Frictional force always opposes the motion or tendency of an object to move against another object or against a surface. We distinguish between two kinds of frictional forces - static and kinetic - because it is observed that kinetic frictional force is slightly less than maximum static frictional force.

Let us now perform the following experiment. Put a block on a rough surface and pull it by a force *F* (see figure 1). Since the force *F* has a tendency to move the block, the frictional force acts in the opposite direction and opposes the applied force *F*. All the forces acting on the block are shown in figure 1. Note that I have shown the weight and the normal reaction acting at two different points on the block. I leave it for you to think why should the weight and the normal reaction not act along the same vertical line?



The applied force F, the weight W, the normal reaction of the surface N and the frictional force acting on a block being pulled on a rough surface

Figure 1

It is observed that the block does not move until the applied force *F* reaches a maximum value F_{max}. Thus from F = 0 up to $F = F_{max}$, the frictional force adjusts itself so that it is just sufficient to stop the motion. It was observed by Coulombs that *F max* is proportional to the normal reaction of the surface on the object. You can observe all this while trying to push a table across the room; heavier the table, larger the push required to move it. Thus we can write

$$F_{\max} \propto N$$

$$\text{or} \quad F_{\max} = \mu_s N$$

where μ_s is known as the coefficient of static friction. It should be emphasize again that is the maximum possible value of frictional force, applicable when the object is about to stop, otherwise frictional force could be less than, just sufficient to prevent motion. We also note that frictional force is independent of the area of contact and depends only on N .

As the applied force F goes beyond F , the body starts moving now experience slightly less force compound to. This force is seem to be when is known as the coefficient of kinetic friction. At low velocities it is a constant but decrease slightly at high velocities. A schematic plot of frictional force F as a function of the applied force is as shown in figure 2.

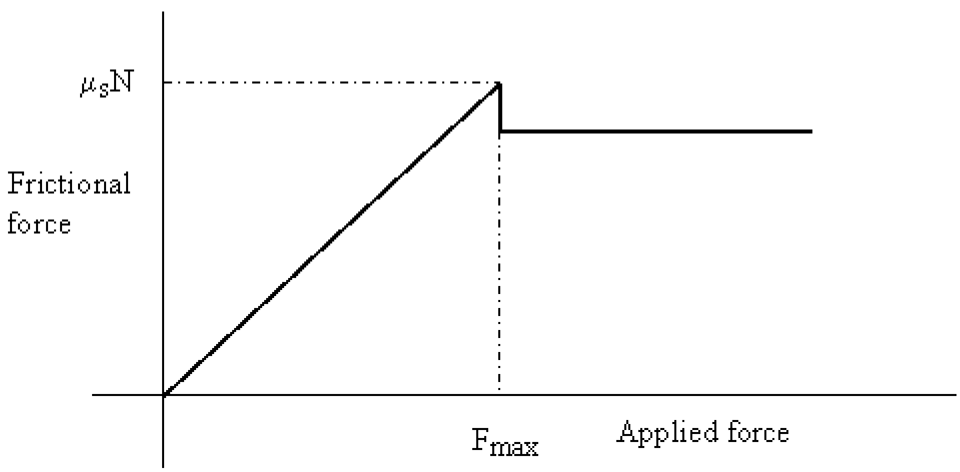
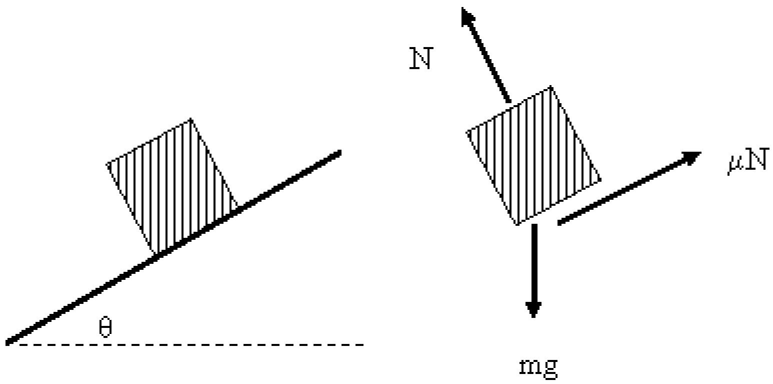


Figure 2

Values of frictional coefficients for different materials vary from almost zero (ice on ice) to as large as 0.9 (rubber tire on cemented road) always remaining less than 1.

A quick way of estimating the value of static friction is to look at the motion an object on an inclined plane. Its free-body diagram is given in figure 3.



A block of mass m on an inclined plane (left) and its free-body diagram (right) when it is about to slide down the ramp

Figure 3

Since the block has a tendency to slide down, the frictional force points up the inclined plane. As long as the block is in equilibrium

$$mg \sin \theta \leq \textit{maximum friction}$$
$$mg \cos \theta = N$$

As θ is increased, mgsinθ increases and when it goes past the maximum possible value of friction f_{max} the block starts sliding down. Thus at the angle at which it slides down we have

$$\left.\begin{aligned} mg \sin \theta = f_{max} &= \mu_s N \\ &= \mu_s mg \cos \theta \end{aligned}\right\} \quad \Rightarrow \quad \mu_s = \tan \theta$$

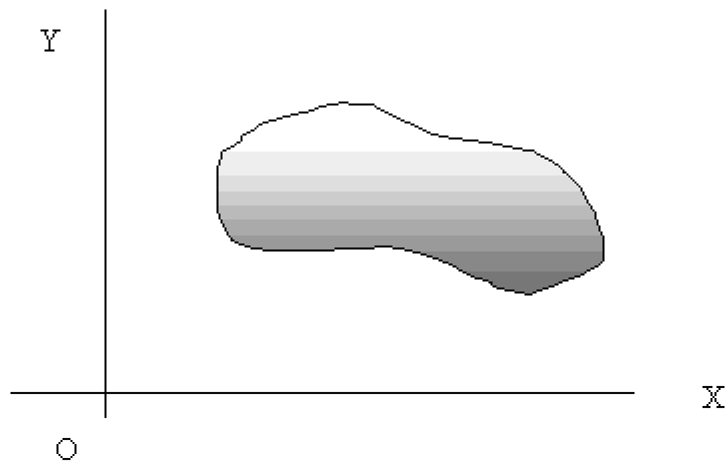
Previous

Lecture 8
Properties of plane surfaces I: First moment and centroid of area

Having deal with trusses and frictional forces, we now change gears and go on to discuss some properties of surfaces mathematically. Of course we keep connecting these concepts to physical situations.

The first thing that we discuss is the properties of surfaces. This is motivated by the fact is general the forces do not act at a single point but are distributes over a body. For example the gravitational force pulling an object down acts over the entire object. Similarly a plate immersed in water, for example has the pressure acting on it over the entire surface. Thus we would like to know at which point does the force effectively act? For example in the case of an object in a gravitational field, it is the centre of gravity where the force acts effectively. In this lecture we develop important mathematical concepts to deal with such forces. Let us start with the *first moment of an area and the centroid* .

First moment of an area and the centroid: We first consider an area in a plane; let us call it the X-Y plane (see figure 1).



An area in the XY plane

Figure 1

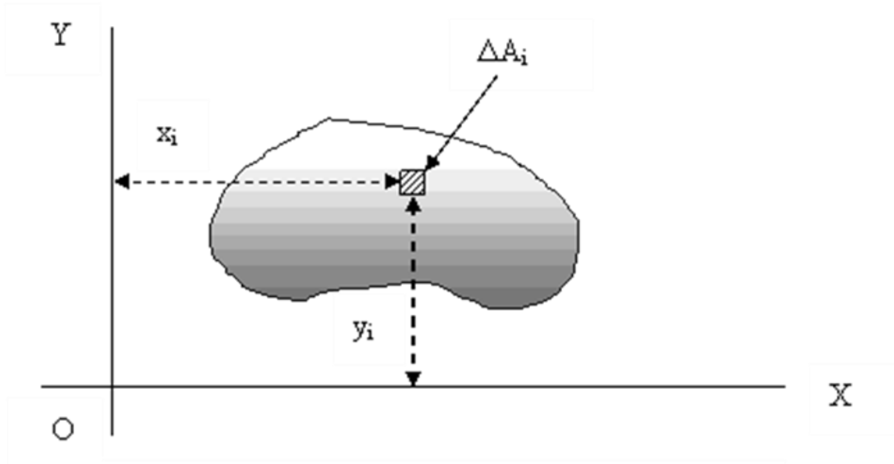
The first moment M_X of the area about the x-axis is defined as follows. Take small area element of area ΔA and multiply it by its y-coordinate, i.e. its perpendicular distance from the X-axis, and then sum over the entire area; the sum obviously goes over to an integral in the continuous limit. Thus

$$M_X = \sum_i y_i \Delta A_i = \int y dA$$

Similarly the first moment M_Y of the area about the y-axis is defined by multiplying the elemental area ΔA by its x-coordinate, i.e. its perpendicular distance from the Y-axis, and summing or integrating it over the entire area. Thus

$$M_Y = \sum_i x_i \Delta A_i = \int x dA$$

This is shown in figure 2.



An element of area ΔA_i and its x- and y-coordinates

Figure 2

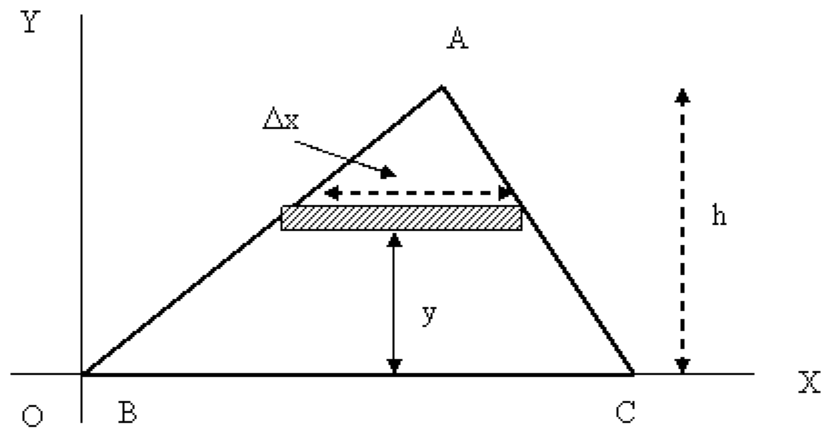
Centroid: Centroid of a bounded area is a point whose x-coordinates X_C and y-coordinate Y_C are defined as

$$X_C = \frac{\int x dA}{A} = \frac{M_Y}{A}$$

$$Y_C = \frac{\int y dA}{A} = \frac{M_X}{A}$$

where *A* is its total area. We now solve some examples of calculating these quantities for some simple areas.

Example 1: We start with the simple example of the first moment and centroid of a triangle with the base along the x-axis. Let its base BC be of length *b*, and let the height of the triangle be *h*. (see figure 3)



An strip of area $\Delta A = (\Delta x)dy$ at height y in triangle ABC

Figure 3

To calculate the M_x, we take a strip of width *dy* at height *y* (see figure 3). Then

$$dA = (\Delta x)dy$$

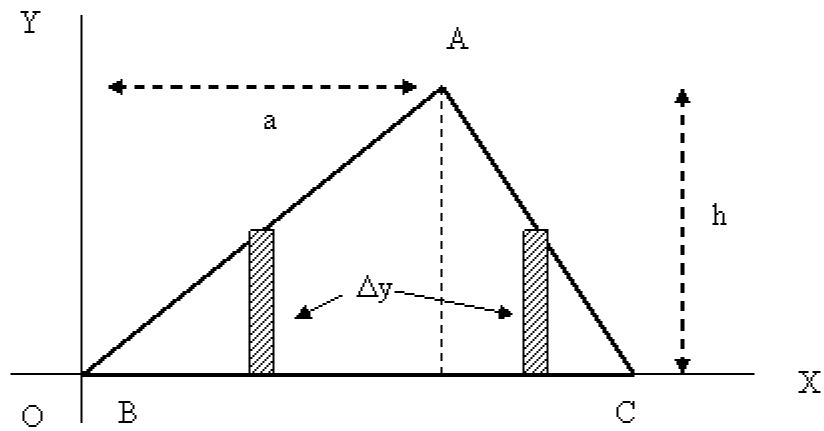
But by similarity of triangles $\frac{\Delta x}{b} = \frac{(h-y)}{h}$. So

$$dA = \frac{b}{h}(h-y)dy$$

Thus

$$M_x = \int_0^h y\,dA = \int_0^h y\frac{b}{h}(h-y)dy = \frac{bh^2}{6} \qquad \Rightarrow \quad Y_c = \frac{M_x}{A} = \frac{h}{3}$$

Let us now calculate the x-co-ordinates for the centroid. For this let the x-coordinate of A be *a* so that the coordinate of point A is (a, h).



Strips of area $\Delta A = (\Delta y)dx$ at distance x in triangle ABC

Figure 4

Now

$$M_y = \int x dA$$

For *dA* let us now take a vertical strip (figure 4). Notice that $(dA = \Delta y dx)$. We will also perform the x-integration in two parts: one from x = 0 to x = a, and the other from x = a to x = b because in the two regions, dependent of *y* on *x* is different so

$$M_y = \int_0^a x\Delta y dx + \int_a^b x\Delta y dx$$

For the region x = 0 to x = a, we can write

$$\frac{\Delta y}{h} = \frac{x}{A} \Rightarrow \Delta y = \frac{h}{A}$$

and for the region x = a to x = b, we have

$$\frac{\Delta y}{(b-x)} = \frac{h}{(b-a)} \quad \Rightarrow \quad \Delta y = \frac{h}{(b-a)}(b-x)$$

Thus

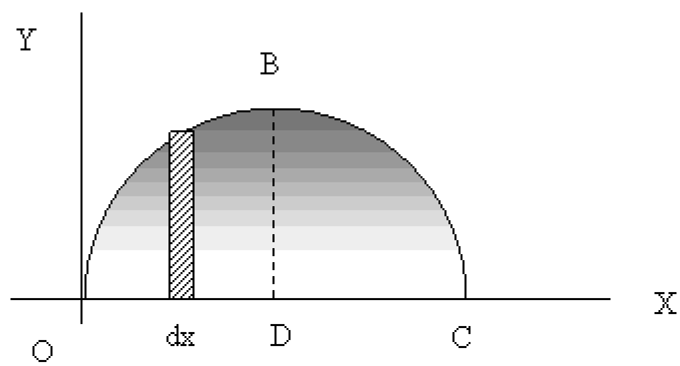
$$M_Y = \int_0^a x\frac{h}{a}x dx + \int_a^b x\frac{h}{(b-a)}(b-x)dx = \frac{hb}{6}(b+a)$$

This gives

$$X_C = \frac{M_y}{A} = \left(\frac{b+a}{3}\right)$$

Thus for a triangle $X_C = \frac{(a+b)}{3}$ and $Y_C = \frac{h}{3}$.

Example2: As the second example, let us calculate the centroid of a semicircular disc of radius *R* . It would be quite easy to solve this problem if the centre *D* of the circle is kept at the origin but I want to do the problem with the disc positioned as drawn below to show you how to tackle the problem.



A semicircular disc of radius R. A vertical strip of width dx at position x is used to determine the x-coordinate of the centroid

Figure 5

The equation of OBC (the circular boundary of the disc) is

$$(x - R^2) + y^2 = R^2$$

where *R* is the radius of the circle. The total area of the plate is $\left(\frac{\pi R^2}{2}\right)$. To calculate X_C , we take a vertical strip of width *dx* at *x* and calculate

$$M_y = \int x dA$$

With $dA = y dx = \sqrt{R^2 - (x-R)^2}\, dx$, we get

$$M_y = \int_0^{2R} x\sqrt{R^2 - (x-R)^2}\,dx$$

To evaluate this integral, we let $x - R = R\sin\theta$ so that the limits of ϑ integration are from $-\frac{\pi}{2}(x=0)$ to $\frac{\pi}{2}(x=2R)$. Then

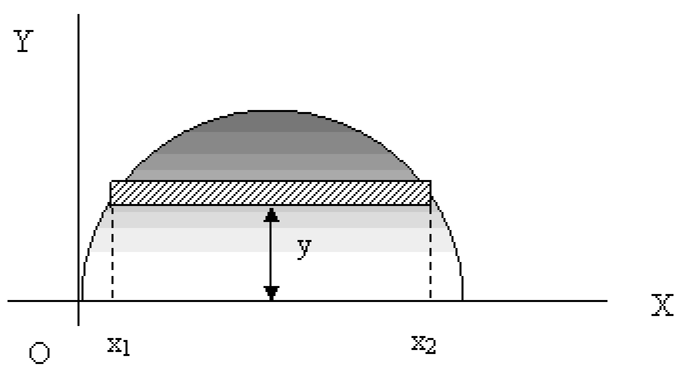
$$M_y = \int_{-\pi/2}^{\pi/2} (R + R\sin\theta).R\cos\theta\, R\cos\theta\, d\theta$$

$$M_y = R^3 \int_{\pi/2}^{\pi/2} \cos^2\theta\, d\theta + R^3 \int_{\pi/2}^{\pi/2} \sin\theta \cos^2\theta\, d\theta \quad = \frac{\pi R^3}{2}$$

which gives

$$X_C = \frac{\pi R^3/2}{\pi R^2/2} = R$$

To calculate Y_C we need to calculate M_X = $\int y dA$, where *dA* represents as strip from x_1 to x_2 (see figure 6)



A semicircular disc of radius R. A horizontal strip of width dy from position x_1 to x_2 is used to determine the y-coordinate of the centroid

Figure 6

From the equation of the circle we get

$$x_1 = R - \sqrt{R^2 - y^2} \quad \text{and} \quad x_2 = R + \sqrt{R^2 - y^2}$$

This gives

$$dA = (x_2 - x_1)dy = 2\sqrt{R^2 - y^2}\,dy$$

and therefore

$$M_Y = \int_0^R 2y\sqrt{R^2 - y^2}\,dy$$

Substituting $y = R \sin \vartheta$, we get

$$M_X = 2\int_0^{\pi/2} R\sin\,\theta.R\cos\theta.R\cos\theta d\theta$$

$$= 2R^3 \int_0^{\pi/2} \cos^2\theta.\sin\,\theta d\theta$$

$$= \frac{2R^3}{3}$$

This gives

$$Y_C = \frac{2R^3}{3\times\frac{\pi R^3}{2}} = \left(\frac{4R}{3\pi}\right)$$

Thus the centroid of the semicircle shown is at $\left(R, \frac{4R}{3\pi}\right)$. Notice that the y coordinate of the centroid is less than $\left(\frac{R}{2}\right)$ which is easily understood because more of the area is concentrated towards the x-axis.

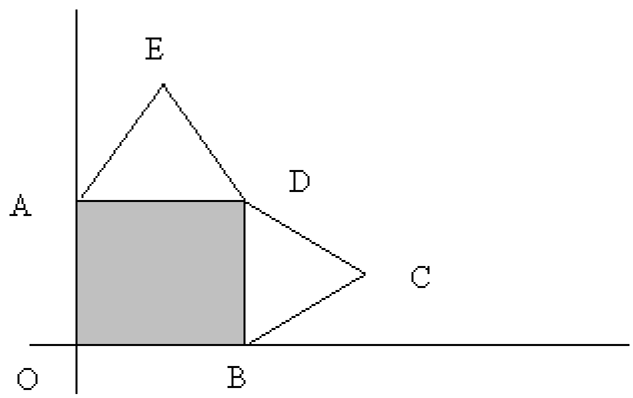
We would no like to emphasize that the centroid *(X_C Y_C)* gives a point fixed in a given planar surface and no matter in which co-ordinate system we calculate this point, it will always come out to be the same point in the surface. Thus it is a property of a surface.

Now let us make one observation: If a body is made up of different shapes of surfaces whose centroid are known. Than the centroid of the composite body

$$X_C = \frac{\sum X_{Ci} A_i}{A_{total}} = \left(\frac{\sum X_{Ci} A_i}{\sum A_i}\right)$$

where X_{Ci} are the centroid of different surfaces and A_i their area. I will leave the simple proof for you, but solve a couple of examples to show you how to use this observation.

Example1: Let us take a square of side *a* and on its two sides let there be two equilateral triangle stuck on it (see figure 7). We wish to calculate the centroid for this surface.



A plane composite surface made by joining square AOBD of side a and two equilateral triangles CDB and EAD

Figure 7

We will consider this body as composed of the square AOBD, the triangle CDE on its right CDE and triangle EAD on its top. Then for the square we have

$$X_{C1}\frac{a}{2},\ Y_{C1}=\frac{a}{2},\ \text{and}\ \ A_1=a^2$$

For the triangle on the right of the square

$$X_{C2}=a+\frac{a}{2\sqrt{3}},\ Y_{C2}\frac{a}{2},\ \text{and}\ \ A_2=\frac{a^2\sqrt{3}}{4}$$

And for the triangle on top of the square

$$X_{C3}=\frac{a}{2},\ Y_{C3}=a+\frac{a}{2\sqrt{3}},\ \ \text{and}\ A_3=\frac{a^2\sqrt{3}}{4}.$$

Thus for the entire plane we get

$$X_C = \frac{X_{C1}A_1 + X_{C2}A_2 + X_{C3}A_3}{A_1 + A_2 + A_3} = \frac{\frac{(11+9\sqrt{3})a^3}{24}}{\frac{(2+\sqrt{3})a^2}{2}} = \frac{(11+9\sqrt{3})a}{12(2+\sqrt{3})}$$

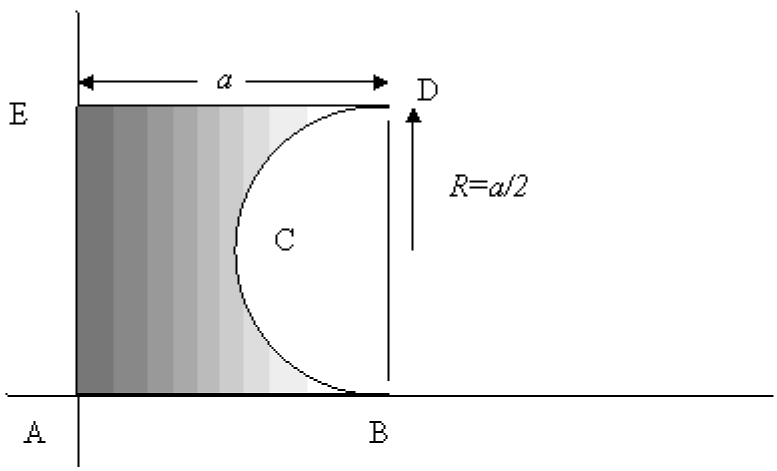
$$= .594a$$

Similarly

$$Y_C = \frac{(11+9\sqrt{3})a}{12(2+\sqrt{3})} = .594a$$

So because of the triangles, the centroid shift a bit to the right and a bit up with respect to the centroid of the square; this happens because of the added area of the triangles.

Example 2: As the second example, let us take an area (ABCDE) that has been obtained by removing a semicircular area from a square. We wish to find its centroid.



Planar surface ABCDE obtained by removing a semicircle from a square

Figure 8

We know the position of the centroid of the square and the semicircular area. Thus

$$A(\text{square}) \times X_C(\text{square}) = A(\text{ABCDE}) \times X_C(\text{ABCDE}) + A(\text{semicircle}) \times X_C(\textit{semicirlce})$$

Therefore

$$X_C(ABCDE) = \frac{A(Square)\,X_C(square) - A(Semicircle)\,X_C(Semicircle)}{A(ABCDE)}$$

From the previous calculation, we know that the centroid for semicircle is

$$\frac{4R}{3\pi} = \frac{4}{3\pi} \times \frac{a}{2} = \left(\frac{2a}{3\pi}\right)$$

from the base. So In the present case we have

$$X_C(Semicircle) = a - \frac{2a}{3\pi}$$

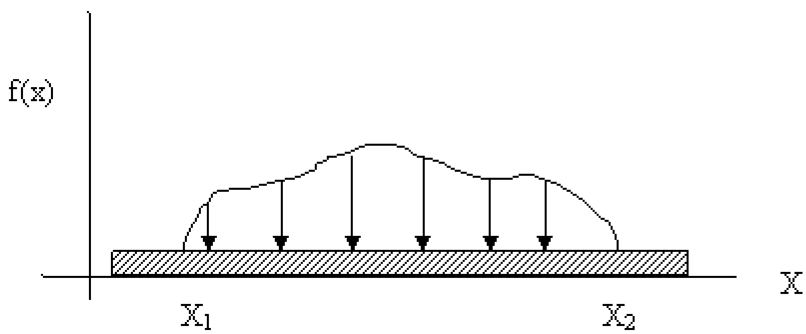
The centroid of the figure *ABCDE* is then

$$X_C(ABCDE) = \frac{a^2 \cdot \frac{a}{2} - \frac{\pi a^2}{8}\left(a - \frac{2a}{3\pi}\right)}{\left(a^2 - \frac{\pi a^2}{8}\right)}$$
$$= .314a$$

which is a little more than *0.25a* . If we had removed a rectangular area equal to half the square, the *X C* for the area left would have been at *0.25a* ; because of the extra area to the right of this point when the semicircle is removed, the centroid shifts slightly to the right.

After introducing you to the mathematical concepts of the first moment and centroid of a surface area, we now apply these ideas to problems in mechanics.

Application to mechanics: As the first simple application of the methods developed let us consider beams which are externally loaded. We consider only those situations where beams are supported externally so that the external reactions can be calculated on the basis of statics alone. As in the case of trusses, such beams are called statically determinate beams. Now one such beam is loaded externally between X_1 and X_2 as shown in figure 9.



A beam loaded externally between X_1 and X_2

Figure 9

In the figure the function f(x) is the load intensity which is equal to load per unit length. Thus force over a length *dx* is given by $dF = f(x)\,dx$. The total load *R* therefore is

$$R = \int_{x_1}^{x_2} f(x)dx = \text{Area under the curve}$$

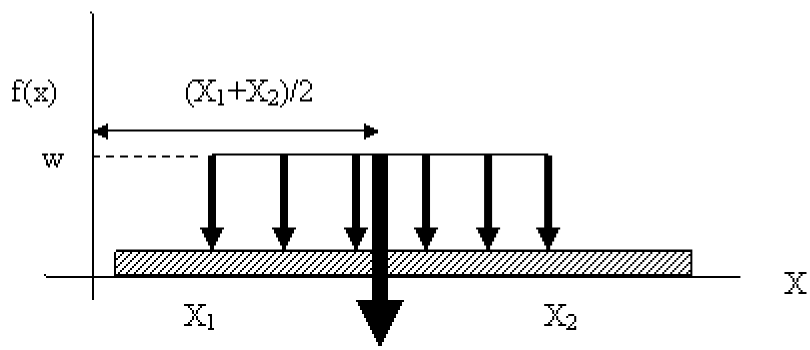
Next question we ask is where is the total load located? This is determined by finding the Moment (torque) created by the load, which is given by

$$\int_{x_1}^{x_2} x f(x) dx = RX$$

$$\text{or } X = \frac{1}{R}\int_{x_1}^{x_2} x f(x) dx$$

Thus the location of the load is given by the centroid of the area formed by the load curve and the beam, taking beam as the x-axis. Let us now take some examples.

Uniform loading: This is shown in figure 10 along with the total load *R* acting at the centroid of the loading intensity curve. The uniform load intensity is *w* .



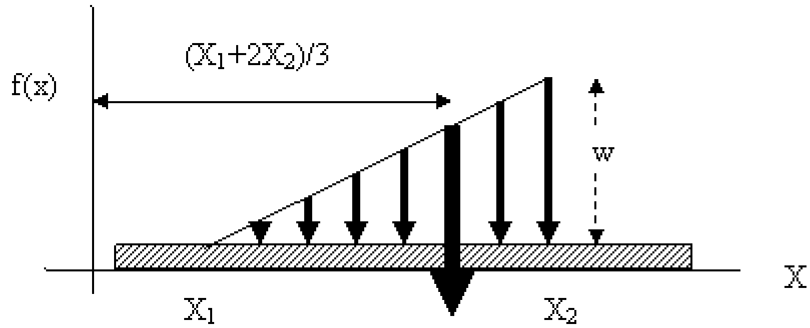
A beam loaded uniformly between X_1 and X_2. Also shown by thick arrow is the total load acting at the centroid of the loading curve

Figure 10

The total load in this case is $R = w(X_2\text{-}X_1)$ and the load acts at the centroid

$$X_c = X_1 + \frac{(X_2 - X_1)}{2} = \frac{(X_1 + X_2)}{2}$$

Triangular loading : This is shown in figure 11 along with the total load *R* acting at the centroid of the loading intensity curve. The height of the triangle is *w* .



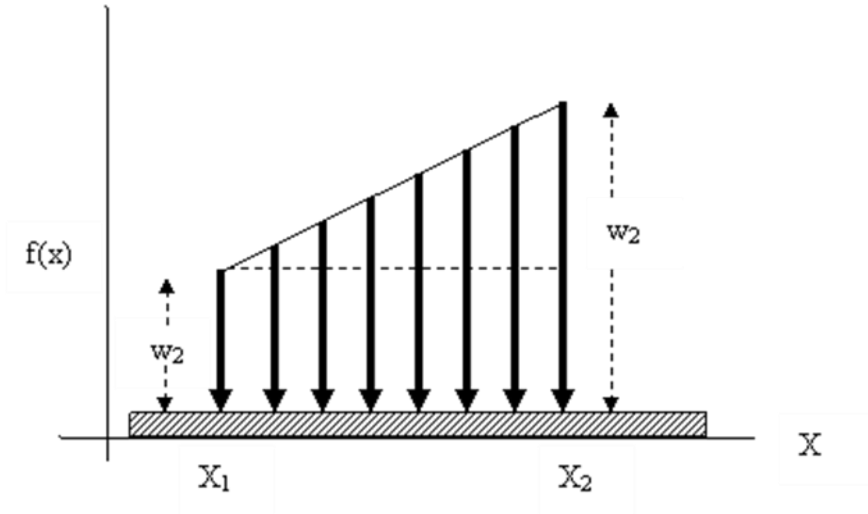
A beam with triangular loading between X_1 and X_2. Also shown by thick arrow is the total load acting at the centroid of the loading curve

Figure 11

In this case the total load is $R = \frac{w(x_1\text{-}x_2)}{2}$ and the load acts at the centroid of the triangle. Recall that for a triangle $X_C = \frac{a+b}{3}$ from the lower left vertex (figure 4) and in the present case $a = (X_2 - X_1)$ and $b = (X_2 - X_1)$. Therefore the centroid is at

$$X_C = X_1 + \frac{2(X_2 - X_1)}{3}$$
$$= \frac{1}{3}(X_1 + 2X_2)$$

I will leave the case of trapezoidal loading (shown in figure 12) for you to work out. You may wish to consider this loading as made up of two different ones: the lower one a rectangular and the upper one a triangular loading.

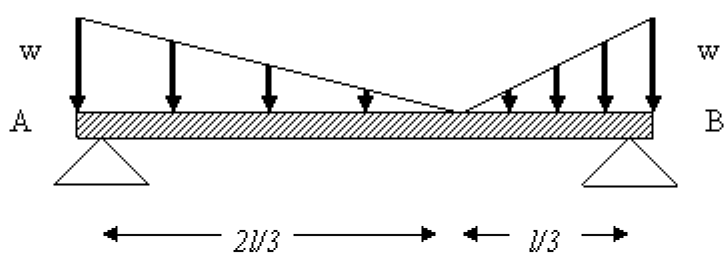


A beam with trapezoidal loading between X_1 and X_2.

Figure 12

Let us now solve an example using here concepts.

Example 3 : In figure 13 a beam on supports A and B is shown with two triangular loadings. All the parameters of the loading are shown in the figure. We wish to know the reaction at supports A and B.



A beam on supports A and B with two triangular loadings. The height of the triangle is w.

Figure 13

If we represent the total loads of the triangles on the left and that on the right by L_1 and L_2 , respectively then

$$L_1 = \frac{1}{2} w \frac{2l}{3} = \frac{wl}{3}$$
$$L_2 = \frac{1}{2} w \frac{l}{3} = \frac{wl}{6}$$

L_1 acts at the centroid of the first triangle which is at a distance

$$X_{C1} = \frac{1}{3}\left(0 + \frac{2l}{3}\right) = \frac{2l}{9}$$

From A. Similarly L_2 acts at the centroid of the second triangle so its distance from A is

$$X_{C2} = \frac{2l}{3} + \frac{1}{3}\left(\frac{l}{3} + \frac{l}{3}\right) = \frac{2l}{3} + \frac{2l}{9} = \left(\frac{8l}{9}\right)$$

Let the reaction at supports A and B be N_A and N_B , respectively. Then

$$N_A + N_B = \frac{wl}{3} + \frac{wl}{6} = \frac{wl}{2}$$

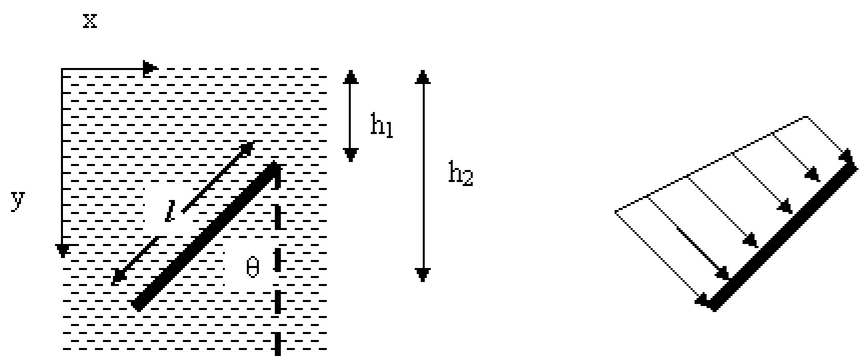
Further, taking moment about A gives

$$\begin{aligned} lN_B &= X_{C1}L_1 + X_{C2}L_2 \\ &= \frac{2l}{9} \times \frac{wl}{3} + \frac{8l}{9} \times \frac{wl}{6} \\ &= \left(\frac{2wl}{9}\right) \quad \Rightarrow \quad N_B = \left(\frac{2w}{9}\right) \end{aligned}$$

Therefore $N_A = \frac{w}{2} - \frac{2w}{9} = \frac{4wl}{18} = \left(\frac{5w}{18}\right)$.

As the next example of the application of the concepts developed, we wish to calculate forces on plane rectangular surfaces submerged in water.

Plane surface Submerged in water: Two questions we wish to answer are (i) What is the average pressure on the plate? and (ii) where does the total force due to the pressure act? Consider a rectangular plate of length *l* and width *w* submerged in water at an angle θ from the vertical as shown in figure 14. The upper end of the plate is at the depth of h_1 and the lower one at the depth of h_2 .



A rectangular plane surface submerged in water (left) and the variation of pressure on it along its length (right)

Figure 14

We first calculate the average pressure on the plate. At a depth y the pressure acting on the plate is $\rho g y$, where ρ is the density of water and g the gravitational acceleration. If we now take a thin strip of width dy at depth y parallel to the plate's width, its area $dA = \frac{w dy}{\cos\theta}$, and the force on it would be $dF = \frac{\rho g y w \, dy}{\cos\theta}$. The total force on the plate would therefore be

$$F = \int \frac{p g y w \, dy}{\cos\theta}$$

This gives the average pressure to be

$$p_{average} = \frac{\dfrac{\int \rho g y w \, dy}{\cos\theta}}{Area \ of \ plate}$$

To understand the significance of the expression above better, let us introduce another length variable Y along the plate (see figure 15).

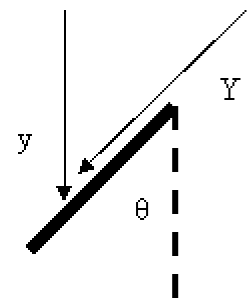


Figure 15

Then we have $Y = \dfrac{y}{\cos\theta}$ so that we can write the average pressure as

$$p_{average} = \frac{\int \rho g Y w \, dY}{Area\ of\ plate} \cos\theta$$

However, $\dfrac{\int Y w \, dY}{Area\ of\ plate} = \dfrac{\int Y \, dA}{Area}$ is the Y-distance of the centroid of the plate. Let us call it Y_C . Thus

$$p_{average} = \rho g Y_C \cos\theta = y_{centroid}$$

Or the average pressure on the plate is *ρg(the depth of the centroid of the plate*). We point out that although we derived this result here for a rectangular plate, the result for the average pressure that $p_{average} = \rho g Y_C \cos\theta = y_{centroid}$ is true for a planar surface of any shape. This is because $p_{average} = \rho g \dfrac{\int Y \, dA}{Area} \cos\theta$ and $\dfrac{\int Y \, dA}{Area}$ is the Y-distance of the centroid of an area of any shape.

Question that we ask now is: at what point does the total force act? To see this let us calculate the moment of the distributed forces due to the pressure. This is given by

$$\int Y dF$$

where

$$dF = \rho g y \, dA$$

For a rectangular plate, $dA = w\,dy = \dfrac{w\,dY}{\cos\theta}$, which is independent of Y. So the loading intensity (force per unit length) for a rectangular plate is going to have the same dependence on the depth as the pressure. Thus the loading on a rectangular plate is trapezoidal as shown in figure 14. The Y-coordinate of the point at which the force acts is

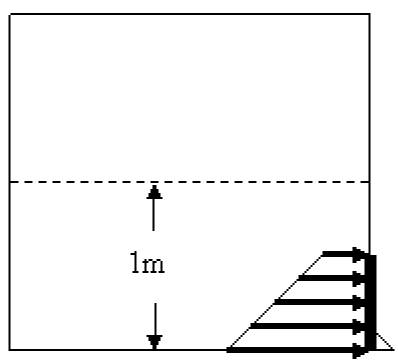
$$\frac{\int Y dF}{\int dF}$$

This by definition is the centroid of the area formed by the loading intensity curve. You have already calculated the centroid of a trapezoidal loading curve. Using that result we find that the total force acts at a depth of

$$\frac{\frac{2}{3}\left(h_1^2 + h_1 h_2 + h_2^2\right)}{\left(h_1 + h_2\right)}$$

Using this result we now solve one example.

Example 4: A two meter high water tank has an opening of the size $\left(\frac{1}{2}m \times \frac{1}{2}m\right)$ at the bottom. The opening is covered by a door hinged on top, shown by A, and is stopped by a fixed wedge, shown by B, at the bottom (see figure 16). Calculate the force on A and B (a) when tank has water filled up to 1m, and (b) has 25cm of water in it. Weight of the door is 19.6N.



Water tank with an opening at the bottom. The door on the opening is also shown along with the loading on it due to water pressure when water is filled up to a height of 1m

Figure 16

(a) As derived earlier, the average pressure on the door will be given by the depth of its centroid. The centroid of the square is at a depth of (0.5 + 0.25 = 0.75m) from the surface of the water. Thus the average pressure is

$$\begin{aligned} p_{average} &= 10^3 \times 9.8 \times 0.75 \\ &= 7350 \, N/m^2 \end{aligned}$$

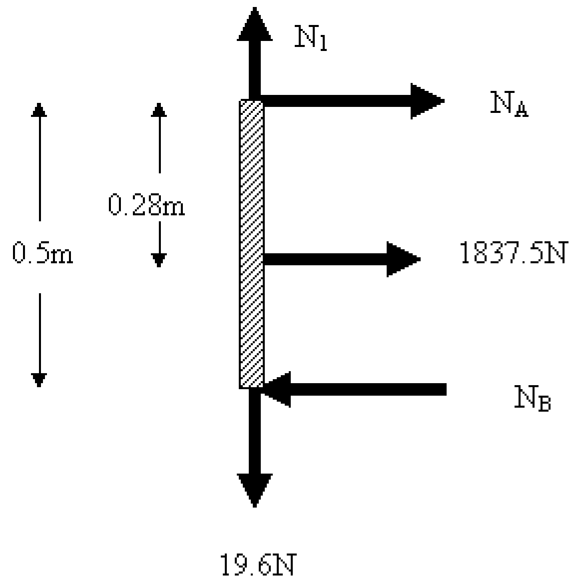
Thus the total force is

F=7350 × .25m 2 =1837.5 N

Notice that having derived our general result for the average pressure earlier, we do not have to perform any integration again to calculate the total force; it is simply the average pressure times the total area. The force is acting at a depth of

$$\frac{2}{3}\frac{(h_1^2 + h_1 \times h_2 + h_2^2)}{(h_1 + h_2)}$$

In the present case $h_1 = 0.5m$, $h_2 = 1.0m$. This gives that the total force is acting at a depth of $\frac{2}{3}\left(\frac{.25+.5+1}{1.5}\right) = \frac{2}{3}\left(\frac{1.75}{1.5}\right) = 0.78m$ or *0 .28m* below A. Thus the free body diagram of the door looks as follows



Free body diagram of the door in example 3

Figure 17

From the force balance equations we have

$N_A + N_B = 1837.5$

and

$N_1 = 19.6N$

The torque balance equation, on the other hand, gives

$.5N\ B = .28 \times 1837.5$

This leads to $N_B = 1029$ N . Putting this in the force balance equation above gives $N_A = 805.5$ N. Thus all the forces have been calculated.

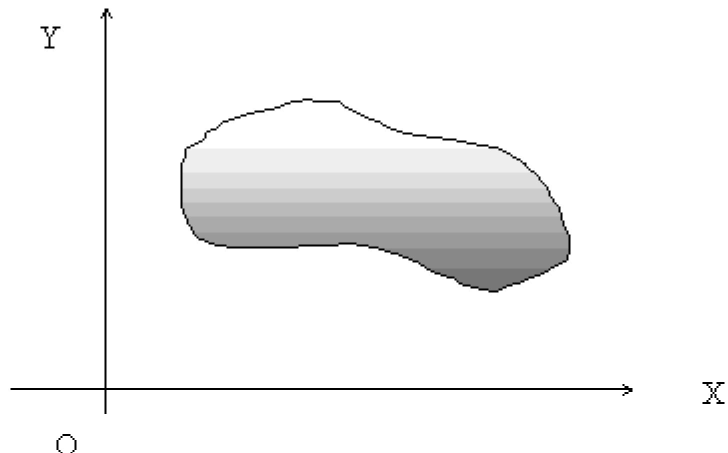
(b) In the second case, the pressure works only on a part of the door and the loading due to the pressure is triangular. Having given this lead to the solution, I'll leave the rest of the problem for you to work out. The answers are $N_A = 25.425$ *N,* $N_B = 127.7$ N and $N_1 = 19.6$ N.

To summarize, in this lecture we have looked at some properties a plane and used it in statics problems. In the next lecture we will expand on this and develop concepts of moment of area and products of area etcetera.

Lecture 9

Properties of surfaces II: Second moment of area

Just as we have discussing first moment of an area and its relation with problems in mechanics, we will now describe second moment and product of area of a plane. In this lecture we look at these quantities as some mathematical entities that have been defined and solve some problems involving them. The usefulness of related quantities, called the moments of inertia and products of inertia will become clear when we deal with rotation of rigid bodies.



An area in the XY plane

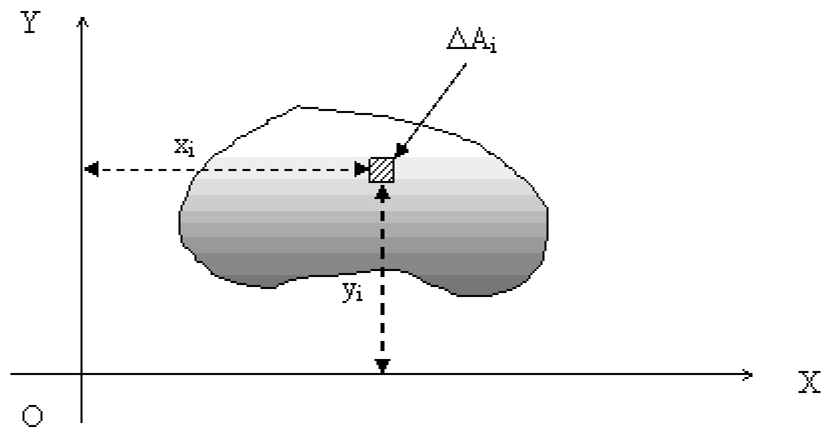
Figure 1

Let us then consider a plane area in xy plane (figure 1). The second moments of the area A is defined as

$$I_{XX} = \sum_i y_i \Delta A_i = \int_A y^2 dA$$

$$I_{YY} = \sum_i x_i \Delta A_i = \int_A x^2 dA$$

That is given a plane surface, we take a small area in it, multiply by its perpendicular distance from the x-axis and sum it over the entire area. That gives I_{XX} . Similarly I_{YY} is obtained by multiplying the small area by square of the distance perpendicular to the y-axis and adding up all contributions (see figure 2).



An element of area ΔA_i and its x- and y-coordinates

Figure 2

The product of area is defined as

$$I_{xy} = \sum_i x_i y_i \Delta A_i = \int xy\, dA$$

where *x* and *y* are the coordinates of the small area *dA* . Obviously I_{XX} is the same as I_{XY} .

Let us now solve a few examples.

Example1: Let us start with a simple example of a square of side a with its center of the origin (see figure 2).

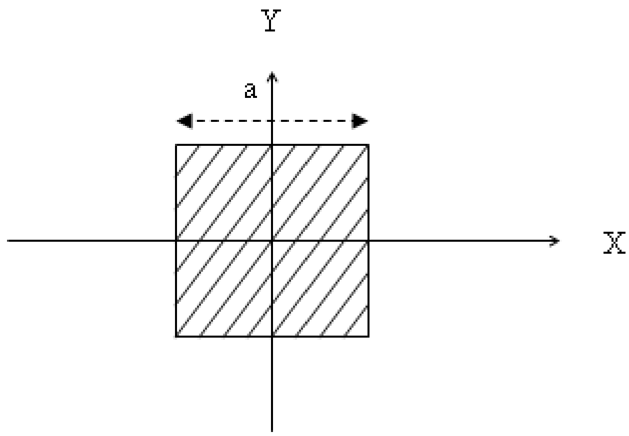


Figure 3

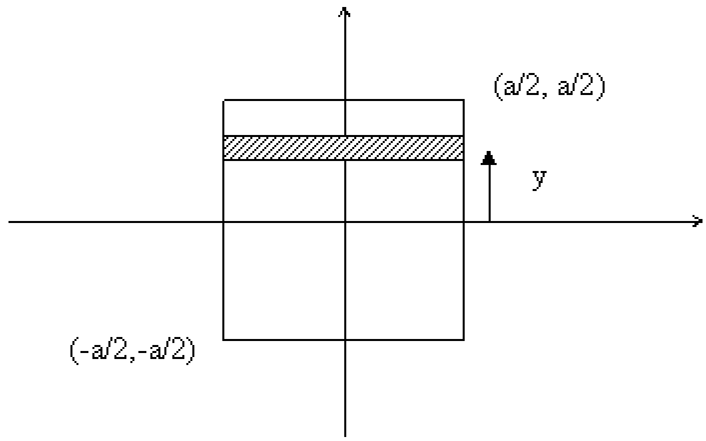
$$I_{XX} = \int_A y^2 dA$$

To calculate this, we choose the elemental area as shown in figure 4 and integrate. Then

dA = ady

so that

$$I_{xx} = \int_{-a\backslash 2}^{a\backslash 2} y^2 a dy = \left(\frac{a^4}{12} \right)$$



Elemental area for calculating I_{XX}

Figure 4

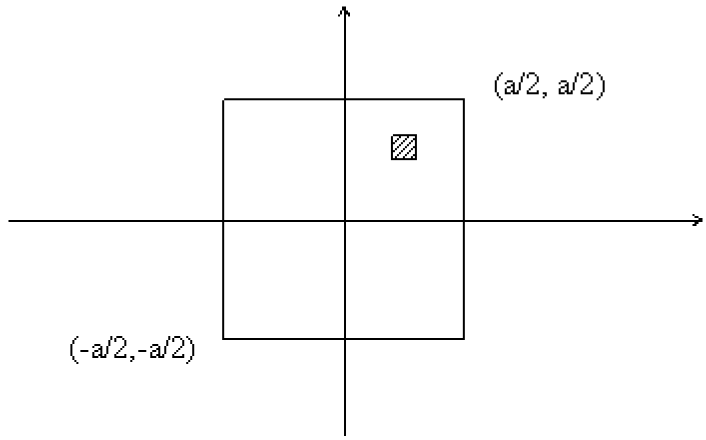
Similarly for calculating I_{YY} we choose a vertical elemental area and calculate

$$I_{yy} = \int_{a\backslash 2}^{a\backslash 2} x^2 a dx = \left(\frac{a^4}{12} \right)$$

Let us also calculate the product of inertia. Choose on elemental area *dxdy* and calculate (see figure 5)

$$I_{xy} = \int_{-a\backslash 2}^{a\backslash 2} x dx \int_{-a\backslash 2}^{a\backslash 2} y dy = 0$$

As noted earlier, I_{YX} is equal to I_{XY} and therefore it also vanishes.

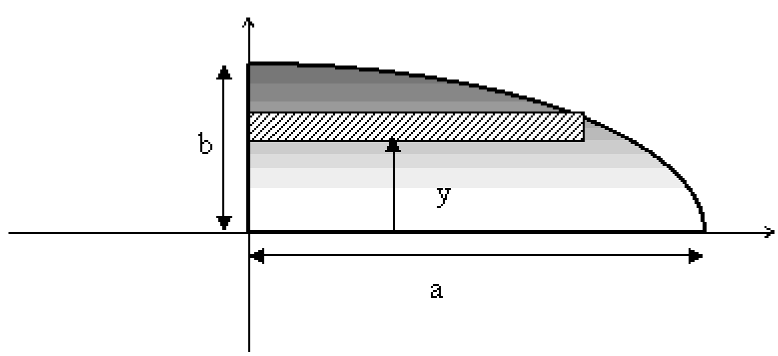


Elemental area dxdy at point (x,y) for calculating I_{XY}

Figure 5

A related problem is that of a rectangular area of size a x b. Its length of side *a* is parallel to the x-axis and the other side of length *b* is parallel to the y-axis. I leave this as an exercise for you to show that in this case $I_{XX} = \left(\frac{ab^3}{12}\right)$, $I_{YY} = \left(\frac{ba^3}{12}\right)$ and $I_{xy} = 0$. Notice that due to the area being symmetrically distributed about the x- and y-axes, the product of the area comes out to be zero.

Example 2 : Next let us consider a quarter of an ellipse as shown in figure 6 and calculate the moment and product of area for this area.



Elemental area for calculating I_{XX} for a quarter of an ellipse

Figure 6

Equation of the ellipse whose quarter is shown in figure 6 is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. For calculating $I_{xx}=\int y^2 dA$ choose an area element parallel to the x-axis to calculate *dA=xdy* and perform the integral

$$I_{XX}=\int y^2 dA=\int_0^b y^2 x dy$$

Using the equation for ellipse, we get

$$x=\sqrt{a^2\left(1-\frac{y^2}{b^2}\right)}=\frac{a}{b}\sqrt{b^2-y^2}$$

which gives

$$I_{XX}=\frac{a}{b}\int_0^b y^2\sqrt{b^2-y^2}\,dy$$

This integral can easily be performed by substituting *y = b* sin *ϑ* and gives

$$I_{XX}=\left(\frac{\pi ab^3}{16}\right)$$

Similarly by taking a vertical strip to perform the integral, we calculate

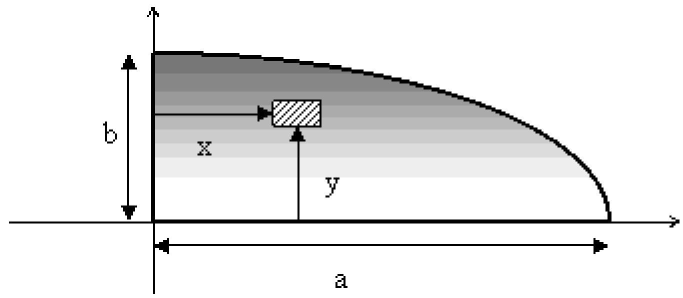
$$I_{YY}=\int x^2 dA=\int_0^a x^2 y\,dx$$

and get

$$I_{YY}=\left(\frac{\pi a^3 b}{16}\right)$$

Next we calculate the product of area I_{XY} . To calculate I_{XY} , we take a small element ($\Delta x\Delta y$) as shown in figure 7, multiply it by *x* and *y* and integrate to get

$$I_{XY}=\int xy\,dx\,dy$$



Elemental area for calculating I_{XY} for a quarter of an ellipse

Figure 7

For a given x , the value of y changes from 0 to $\frac{b}{a}\sqrt{a^2 - x^2}$ so the integral is

$$I_{XY} = \int_0^a x\,dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy$$

This integral is easily performed to get

$$I_{XY} = \left(\frac{b^2a^2}{8}\right)$$

Thus for a quarter of an ellipse, the moments and products of area are

$$I_{XX} = \frac{\pi a^3 b}{16},\ I_{YY} = \frac{\pi a^3 b}{16} \text{ and } I_{XY} = \frac{b^2 a^2}{8}$$

If we put $a = b$, these formulas give the moments and products of area for a quarter of a circle of radius a . I will leave it for you to work out what will be I_{XX} , I_{YY} and I_{XY} for the full ellipse about its centre.

Using the second moment of an area, we define the concept of the radii of gyration. This is the point which will give the same moment of inertia as the area under consideration if the entire area was concentrated there. Thus

$$Ak_X^2 = I_{XX} = \int y^2 dA, \quad \text{and} \quad Ak_Y^2 = I_{YY} = \int x^2 dA$$

define the radii of gyration k_X and k_Y about the x- and the y-axes, respectively. In the example of a rectangular area of size *a x b* with side a parallel to the x-axis, we had $I_{XX} = \left(\frac{ab^3}{12}\right)$, $I_{YY} = \left(\frac{ba^3}{12}\right)$. So for this rectangle, the radii of gyration are $k_X = \frac{b}{2\sqrt{3}}$ and $k_Y = \frac{a}{2\sqrt{3}}$.

Having defined the moments and products of area, we now describe a relationship between the second moment of an area about a set of axes passing through the centroid of that area and another set of (x-y) axes which are parallel to those passing through the centroid. This is known a transfer theorem.

Transfer theorem: Let the centroid of an area be at point ($x_0\ y_0$) with respect to the set of axes (*xy*). Let ($x'\ y'$) be a parallel set of axes passing through the centroid. Then

$$\begin{aligned} I_{XX} &= \int y^2 dA = \int (y' + y_0)^2\, dA \\ &= \int y'^2\, dA + \int y_0^2 dA + 2y_0 \int y' dA \end{aligned}$$

But by definition

$$\begin{aligned} \int y' dA &= A \times (y' \text{ coordinate of centroid in } x'y') \\ &= A \times 0 = 0 \end{aligned}$$

which gives

$$I_{XX} = I_{X'X'} + y_0^2 A$$

This is how the moment of area of a plane about an axis is related to the moment of the same area about another axis parallel to the previous one but passing through the centroid. Similarly it is easily shown that

$$I_{YY} = I_{Y'Y'} + x_0^2 A.$$

and

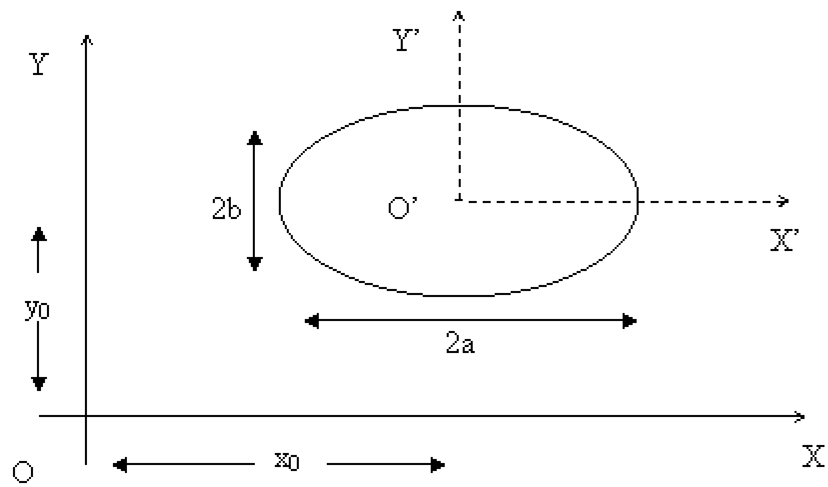
$$I_{XY} = I_{X'Y'} + x_0 y_0 A$$

We now solve an example to show the application of this theorem.

Example 3 : Calculate the second moments and products of area of an ellipse with its centre at (x_0, y_0)

In a previous exercise, you have already calculated the second moments and products of area of an ellipse about its centre, which is also its centroid. These are:

$$I_{XX}(ellipse) = \frac{\pi a^3 b}{4} \quad ; \quad I_{YY}(ellipse) = \frac{\pi a b^3}{4} \quad \text{and} \quad I_{XY}(ellipse) = 0$$



An ellipse with its centre at (x_0,y_0)

Figure 8

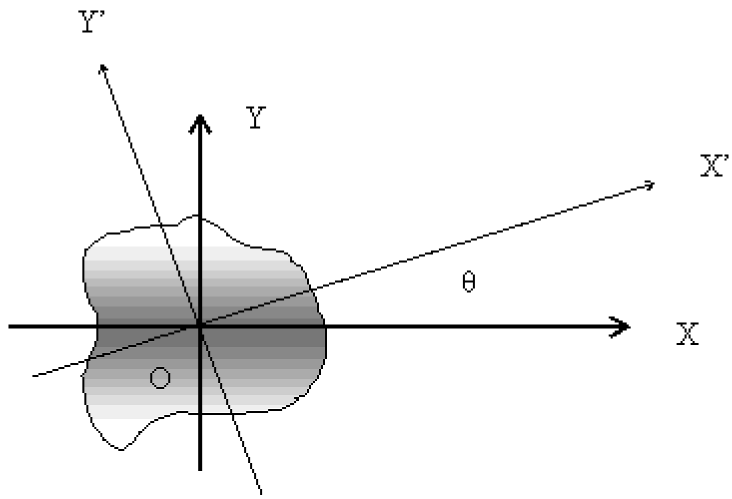
We use these results now in applying the transfer theorem to obtain moments and products of area of the ellipse about a different origin (see figure 8) . Thus

$$I_{XX}\,(about\ O) = (Area\ of\ ellipse)\ x_0^2 + I_{X'X'}(about\ centroid)$$
$$= \pi ab x_0^2 + \frac{\pi a^3 b}{4}$$

Similarly

$$I_{YY}\,(about\ O) = \pi ab y_0^2 + \frac{\pi a b^3}{4}$$
$$I_{XY}\,(about\ O) = \pi ab x_0 y_0 + 0 = \pi ab x_0 y_0$$

Transformation of moments and products of area from one system to another rotated with respect to the first one: We just learnt that if we translate an area so that its centriod moves to another point, how its second moments of inertia and products of inertia change when the axes passing through the centroid and the other set of axes are parallel. We now study how the moments and products are related when we calculate them about another set of axes that an rotated with respect to the first one. So we consider a set of area (xy) and another on (x'y') rotated with respect to the first one by an angle θ (see figure 1).



An object viewed from two different set of axes rotated with respect to one another

Figure 9

We wish to relate $I_{X'X'}$, $I_{Y'Y'}$ and $I_{X'Y'}$ to $I_{XX}, I_{YY}\ and\ I_{XY}$. In lecture 1, we have already learnt that

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

This gives

$$I_{x'x'} = \int y'^2 dA = \int x^2 \sin^2\theta dA + \int y^2 \cos^2\theta dA - \int 2xy\sin\theta\cos\theta dA$$

$$= I_{yy}\sin^2\theta + I_{xx}\cos^2\theta - I_{xy}\sin^2\theta$$

changing $\sin^2\theta = \frac{1}{2}(1-\cos^2\theta)$ and $\cos^2\theta = \frac{1}{2}(1+\cos^2\theta)$ we get

$$I_{x'x'}' = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

Similarly

$$I_{y'y'} = \int x'^2 dA = \int x^2 \cos^2\theta\, dA + \int y^2 \sin^2\theta\, dA + \int 2xy \sin\theta \cos\theta dA$$

$$= I_{yy} \cos^2\theta + I_{xy} \sin^2\theta - I_{xy} \sin^2\theta$$

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\theta + I_{xy} \sin 2\theta$$

and

$$I_{x'y'} = \int x'y'\, dA$$

$$= \int (x\cos\theta + y\sin\theta) + \int (-x\sin\theta + y\cos\theta) dA$$

$$= -\int (x^2 \cos\theta \sin\theta + \int y^2 \sin\theta \cos\theta dA + \int xy(\cos^2\theta \sin^2\theta) dA$$

$$= -\frac{I_{yy}\sin 2\theta}{2} + \frac{I_{xx}\sin 2\theta}{2} + I_{xy}\cos\theta$$

Thus

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

This gives the second moment and product about a set of axis (x'y') rotated about the other set (xy). Let us now discuss some examples.

As expected for a circular area, no matter about which set of axes you calculate I_{xx}, I_{yy}, I_{xy}, it will always come out to be the same because the area looks the sum from all set axes. What is interesting, however, is that for a square also the moments and product of area are the same with respect to any set of axes passing through its centre. It happens because with respect to its centre, the I_{XX} and I_{YY} for a square are the same i.e. $I_{xx} = I_{yy} = \frac{a^4}{12}$ and $I_{xy} = 0$. This is left as an exercise for you to show.

We now use for formulae derived above to obtain what we call the *principal set of axes* for a plane area. The principal set of axes at a point are those for which the product of inertia vanishes i.e. about the principal set of axes $I_{xy} = 0$. Let us see how we determine these axes if we know I_{xx}, I_{yy} and I_{xy} about a given set of axes. In the following we refer to the principal set of axes as *(1,2)* where *1* refers to the x-axis and *2* to the y-axis. We know that we want

$$I_{12} = \frac{I_{xx} - I_{yy}}{2} \sin 2\alpha + I_{xy} \cos 2\alpha = 0$$

where α is the angle the principal set of axes make with the (xy) set of axes. The equation above gives

$$\tan 2\alpha = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

The principal set of axes has one more property: The moments of area is maximum one of the principal axis (say x-axis) and minimum about the other (y-axis). This is seen as follow: Since

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

Let us find θ for which $I_{X'X'}$ is a maximum or a minimum. The condition

$$\frac{\partial I_{xx'}}{\partial \theta} = 0$$

gives

$$-(I_{xx} - I_{yy}) \sin 2\theta - 2I_{xy} \cos 2\theta = 0$$
$$\text{or } \tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

This is the same angle a that makes I_{XY} vanish. This means

$$2\theta = 2\alpha \text{ or } 2\alpha + \pi$$

Thus

$$\theta = \alpha \text{ or } (\alpha + \frac{\pi}{2})$$

When α makes the function I_{XX} a maximum, the angle $\left(\alpha + \frac{\pi}{2}\right)$ makes I_{YY} a minimum. I'll leave it for you to show that. Thus the principal set of axes are also those about which the II nd moment of area is maximum about one axis and minimum about the other. Notice that for a square, any set of axes passing through its centre is a principal set of axes. This follows from the exercise that you did above. As a related quantity, we also define *polar moment of an area* . This is calculated as

$$I = \int r^2 dA = \int x^2 dA + \int y^2 dA = I_{xx} + I_{yy}$$

Since r^2 is independence of the (xy) system chosen, *I* is the same about any set of axes passing through a point.

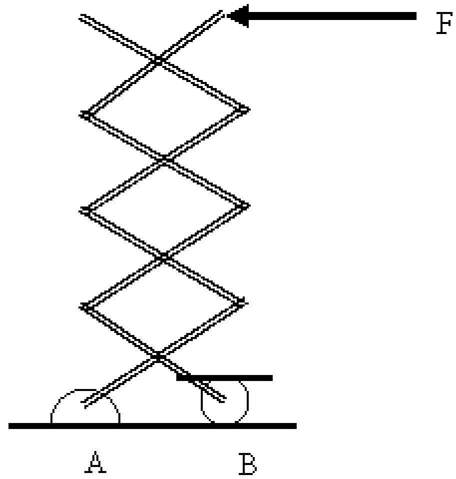
Having defined these concepts, at the end I will point out that in a similar manner IInd moment of mass can also be defined. We will elaborate on that more in the later lectures on dynamics when we deal with the rotation of rigid bodies.

Lecture 8 and 9 conclude our introduction to the properties of surfaces.

Lecture 10
Method of Virtual Work

So far when dealing with equilibrium of bodies/trusses etcetera, our strategy has been to isolate parts of the system (subsystem) and consider equilibrium of each subsystem under various forces: the forces that we apply on the system and those that the surfaces, and other elements of the system apply on the subsystem. As the system size grows, the number of subsystems and the forces on them becomes very large. The question is can we just focus on the force applied to get it directly rather than going through each and every subsystem. The *method of virtual work* provides such a scheme. In this lecture, I will give you a basic introduction to this method and solve some examples by applying this method.

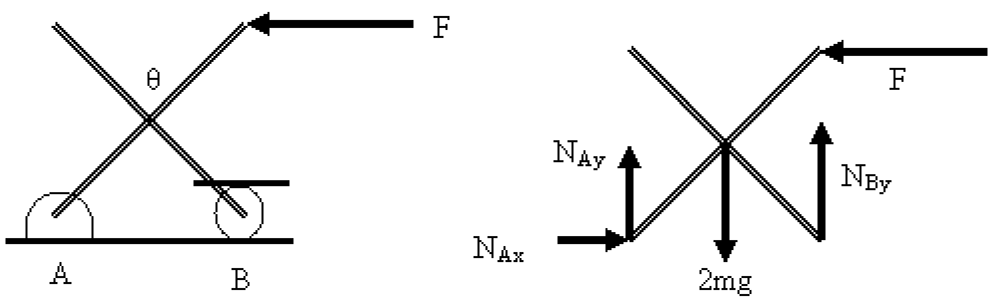
Let us take an example: You must have seen a children's toy as shown in figure 1. It is made of many identical bars connected with each other as shown in the figure. One of the lowest bars is connected to a fixed pin joint *A* whereas the other bar is on a pin joint *B* that can move horizontally. It is seen that if the toy is extended vertically, it collapses under its own weight. The question is what horizontal force *F* should we apply at its upper end so that the structure does not collapse.



A folding toy made up of many bars in equilibrium under an applied force F

Figure 1

To see how many equations do we have to solve in finding *F* in the structure above, let us take a simple version of it, made up of only two bars, and ask how much force F do we need to keep it in equilibrium (see figure 2).



Two identical bars of length l and mass m each. Support A is a fixed pin joint whereas support B, also a pin joint, is free to move horizontally (left). The free-body diagram is shown on the right.

Figure 2

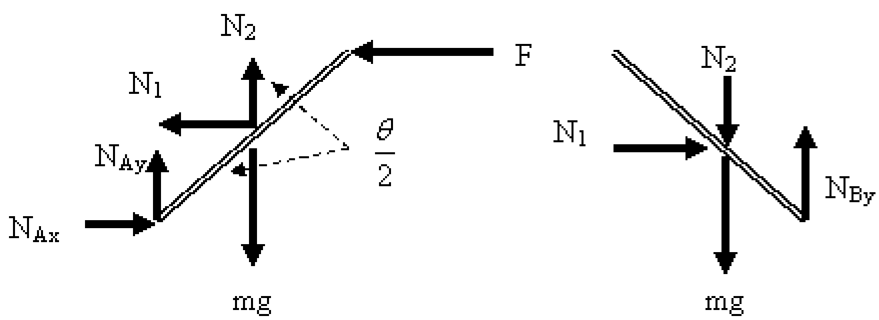
Let each bar be of length l and mass m and let the angle between them be θ. The free-body diagram of the whole system is shown above. Notice that there are four unknowns - N_{Ax} , N_{Ay} , N_{By} and F - but only three equilibrium equations: the force equations

$$N_{Ay} + N_{By} = 2mg \quad \text{and} \quad N_{Ax} + F = 0$$

and the torque equation

$$2\text{mg} \times \frac{1}{2}\sin\frac{\theta}{2} - N_{By} \times l\sin\frac{\theta}{2} - F \times l\cos\frac{\theta}{2} = 0$$

So to solve for the forces we will have to look at individual bars. If we look at individual bars, we also have to take into account the forces that the pin joining them applies on the bars. This introduces two more unknowns N_1 and N_2 into the problem (see figure 3). However, there are three equations for each bar - or equivalently three equations above and three equations for one of the bars - so that the total number of equations is also six. Thus we can get all the forces on the system.



Free body diagram of the two bars of the structure of figure 2

Figure 3

The free-body diagrams of the two bars are shown in figure 3. To get three more equations, in addition to the three above, we can consider equilibrium of any of the two bars. In the present case, doing this for the bar pinned at B appears to be easy so we will consider that bar. The force equations for this bar give

$$N_1 = 0 \quad \text{and} \quad N_{By} = N_2 + mg$$

And taking torque about B , taking $N_1 = 0$, gives

$$N_2 = -mg$$

This then leads to (from the force equation above)

$$N_{By} = 0$$

Substituting these in the three equilibrium equations obtained for the entire system gives

$$N_{Ay} = 2mg\,, \quad F = mg\tan\frac{\theta}{2} \quad \text{and} \quad N_{Ax} = -mg\tan\frac{\theta}{2}$$

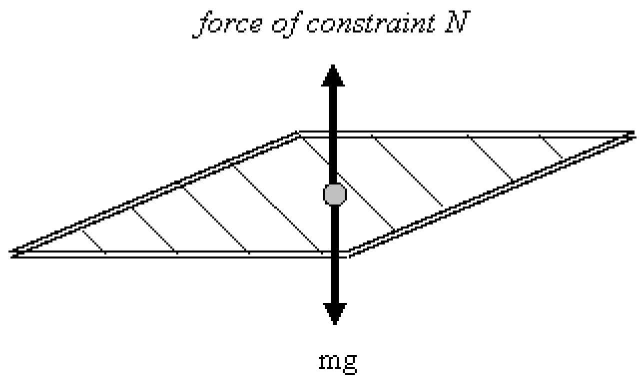
Looking at the answers carefully reveals that all we are doing by applying the force *F* is to make sure that the bar at pin-joint A is in equilibrium. This bar then keeps the bar at joint B in equilibrium by applying on it a force equal to its weight at its centre of gravity.

The question that arises is if we have many of these bars in a folding toy shown in figure 1, how would we calculate *F* ? This is where the *method of virtual work*, to be developed in this lecture, would come in handy. We will solve this problem later using the method of virtual work. So let us now describe the method. First we introduce the terminology to be employed in this method.

1. Degrees of freedom: This is the number of parameters required to describe the system. For example a free particle has three degrees of freedom because we require *x*, *y* , and *z* to describe its position. On the other hand if it is restricted to move in a plane, its degrees of freedom an only two. In the mechanism that we considered above, there is only one degree of freedom because angle θ between the bars is sufficient to describe the system. Degrees of freedom are reduced by the constraints that are put on the possible motion of a system. These are discussed below.

2. Constraints and constraint forces: Constraints and those conditions that we put on the movement of a system so that its motion gets restricted. In other words, a constraint reduces the degrees of freedom of a system. Constraint forces are the forces that are applied on a system to enforce a constraint. Let us understand these concepts through some examples.

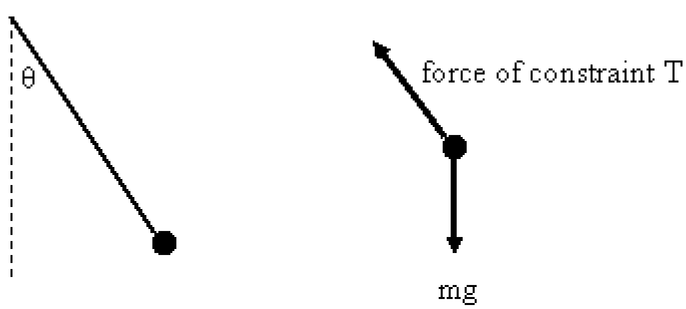
A particle in free space has three degrees of freedom. However, if we put it on a plane horizontal surface without applying any force in the vertical direction, its motion is restricted to that plane. Thus now it has only two degrees of freedom. So the constraint in this case is that the particle moves on the horizontal surface only. The corresponding force of constraint is the normal reaction provided by the surface.



A particle on a horizontal surface has two degrees of freedom with the normal reaction providing the constraint force

Figure 4

As the second example, let us take the case of a vertical pendulum oscillating in a plane (see figure 5). Thus its degrees of freedom would be two if there were no more constraints on its motion. However, the bob of a pendulum is constrained to move in such a way that its distance from the pivot point remains fixed. We have thus introduced one more constraint on its motion and therefore the degrees of freedom are reduced by one; a pendulum oscillating in a plane has only one degree of freedom. The angle from the equilibrium position is therefore sufficient to describe a plane pendulum's motion fully. How about the force of constraint in this case? The constraint, that the distance of the bob from the pivot point remains fixed, is ensured by the tension in the string. The tension in the string is therefore the force of constraint.



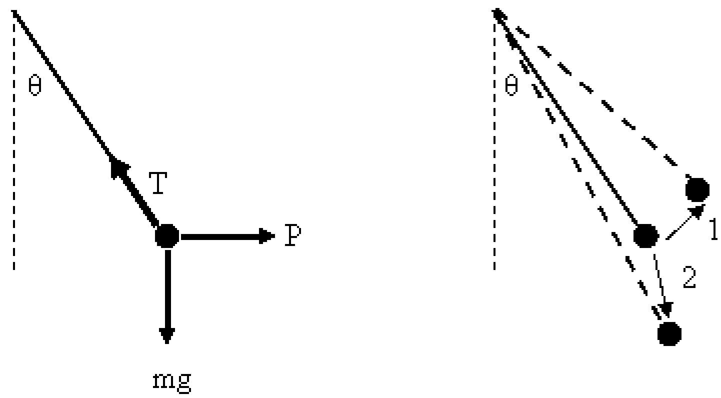
A plane pendulum (left) has only one degree of freedom. The free body diagram (right) of the bob shows the gravitational force and the force of constraint acting on it

Figure 5

Let us now consider the folding toy shown in figure 1. This structure, although made of many moving bars, has only one degree of freedom because the bars are constrained to move in a very specific way. Thus from a large number of degrees of freedom for these bars, all of them except one are eliminated by the constraints. As such the number of constraints, and therefore the number of constraint forces, is very large. The constraint forces are the reactions at the supports A and B and the forces applied by the pins holding the bars together. It is because of these forces that the system is restricted in its motion.

I would like you to note one thing interesting in the examples considered above: if the system moves the constraint forces do not do any work on it. In the case of a particle moving on a plane, the motion is perpendicular to the normal reaction so it does no work on the particle. In the pendulum the motion of the bob is also perpendicular to the tension in the string which is the force of constraint. Thus no work is done on the bob by the constraint force. The case of the toy in figure 1 is quite interesting. In the structure point A does not move and the motion of point B is perpendicular to the reaction force at B. Thus there is no work done by the reaction forces at these points. On the other hand, the constraint forces due to pins connecting two bars are equal and opposite on each bar. But the points on the bar where these forces act (the points where the pin joints are) have the same displacement for each bar so that the net work done by the constraint forces vanishes.

3. Virtual displacement: Given a system in equilibrium, its virtual displacement is *imagined* as follows: Move the system slightly away from its equilibrium position arbitrarily but consistent with the constraints. This represents a virtual displacement of the system. Note the emphasis on the word imagined. This is because a virtual displacement is not caused by the applied forces. Rather it is the difference between the equilibrium position of the system and an imagined position - consistent with the constraints - of the system slightly away from the equilibrium. For example in the case of a pendulum under equilibrium at an angle θ under a force P (see figure 6), virtual displacement would be increasing the angle from θ to $(\theta + \Delta\theta)$ keeping the distance of the bob from the pivot unchanged. On the other hand, moving the bob with a component in the direction of the string is not a virtual displacement because it will not be consistent with the constraint. Virtual displacement is denoted by $\delta\vec{r}$ to distinguish it from a real displacement $d\vec{r}$.



A plane pendulum (left) in equilibrium under force P. Two possible displacements of the bob are shown on the right. The displacement shown by arrow 1 is a virtual displacement but that shown by arrow 2 is not.

Figure 6

4. Virtual work: The work done by any force $\vec{F}$ during a virtual displacement is called virtual work. It is denoted by δW . Thus

$$\delta W = \vec{F} \cdot \delta\vec{r}$$

Note that our previous observation, that work done by a constraint force is usually zero, implies that virtual work done by a constraint force is also zero. Also keep in mind that in calculating the work $\vec{F} \cdot \delta\vec{r}$ done by the force $\vec{F}$, $\delta\vec{r}$ represents the displacement of the point where the force is being applied.

With these definitions we are now ready to state the principle of virtual work. It is based on the assumption that virtual work done by a constraint force is zero. The principle of virtual work states that " *The necessary and sufficient condition for equilibrium of a mechanical system without friction is that the virtual work done by the externally applied forces is zero* ". Let us see how it arises. For a system in equilibrium, each particle in the system is in equilibrium under the influence of externally applied forces and the forces of constraints. Then for the i^{th} particle

$$\vec{F}_{external,i} + \vec{F}_{constraint,i} = 0 \Rightarrow \left(\vec{F}_{external,i} + \vec{F}_{constraint,i}\right) \cdot \delta\vec{r}_i = 0$$

Therefore

$$\sum_i \left(\vec{F}_{external,i} + \vec{F}_{constraint,i}\right) \cdot \delta\vec{r}_i = 0$$

But we have already seen that for individual particles $\vec{F}_{constraint,i} \cdot \delta r_i = o$ and for a system composed of many subsystems $\sum_i \vec{F}_{constraint,i} \cdot \delta r_i = 0$, that is the net virtual work done by constraint forces is zero. This means that the total virtual work done by the external forces vanishes, i.e.

$$\sum_i \vec{F}_{external,i} \cdot \delta\vec{r}_i = 0$$

This is the necessary part of the proof. The condition is also sufficient condition. This is proved by showing that if the body is not in equilibrium, the virtual work done by the external forces does not vanish for all arbitrary virtual displacements (consistent with the constraints). If the body is not in equilibrium, it will move in the direction of the net force on each particle. During this real displacement $d\vec{r}_i$ the work don by the force on the ith particle will be positive i.e.

$$\left(\vec{F}_{external,i} + \vec{F}_{constraint,i}\right) \cdot d\vec{r}_i > 0$$

Now we can choose this real displacement to be the virtual displacement and find that when the body is not in equilibrium, all virtual displacements consistent with the constraints will not give zero virtual work. Thus when the system is not in equilibrium

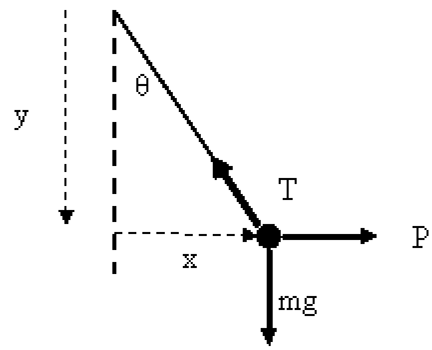
$$\left(\vec{F}_{external,i} + \vec{F}_{constraint,i}\right) \cdot \delta\vec{r}_i > 0 \Rightarrow \sum_i \left(\vec{F}_{external,i} + \vec{F}_{constraint,i}\right) \cdot \delta\vec{r}_i > 0$$

Assuming again that the net work done by the constraint forces is zero, we get that for a body not in equilibrium

$$\sum_i \vec{F}_{external,i} \cdot \delta\vec{r}_i \neq 0$$

This implies that when the virtual work done by external forces vanishes, the system must be in equilibrium. This proves the sufficiency part of the condition. We now solve some examples to illustrate how the method of virtual work is applied.

Example 1: A pendulum in equilibrium as shown in figure 5. We show the coordinates of the bob in the figure 7 below.



A plane pendulum in equilibrium under force P

Figure 7

$x_{bob} = l \sin \theta$

$y_{bob} = l \cos \theta$

If the pendulum is give a virtual displacement i.e. $\theta \to \theta + \Delta \theta$

$\Delta x = l \cos \theta \Delta \theta$
$\Delta y = -l \sin \theta \Delta \theta$

By the principle of virtual work, the total virtual work done by the external forces vanishes at equilibrium. So the equilibrium is described by

$Pl \cos \theta \Delta \theta - mgl \sin \theta \Delta \theta$

giving

$P = mg \tan \theta$

Which is the same answer as obtained earlier.

Example 2: This is the problem involving two crossed bars as shown in figure 2. We wish to calculate the force *F* required to keep the system in equilibrium using the principle of virtual work.

To apply the principle of virtual work, imagine a virtual displacement consistent with the constraint. The only displacement possible - because of only one degree of freedom - is that

$\theta \rightarrow \theta + \Delta\theta$. From figure 2 it is clear that the external forces on the system are *F* and *2mg* (weight of the bars).

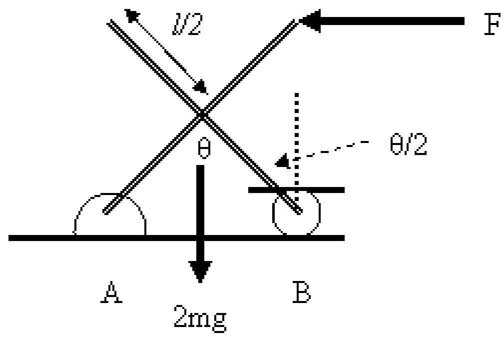


Figure 8

As θ increased to θ + Δθ , the point where the bars cross moves down by a distance (see figure 8)

$$\frac{l}{2}\cos\left(\frac{\theta}{2}\right) - \frac{l}{2}\cos\left(\frac{\theta+\Delta\theta}{2}\right) = \frac{l}{4}\sin\left(\frac{\theta}{2}\right)\Delta\theta$$

and the point when *F* is applied moves to the right by a distance

$$l\sin\left(\frac{\theta+\Delta\theta}{2}\right) - l\sin\left(\frac{\theta}{2}\right) = \frac{l}{2}\cos\left(\frac{\theta}{2}\right)\Delta\theta$$

To calculate the net virtual work done, I remind you that work by a force $\vec{F}$ is calculated by taking the dot product $\vec{F}\cdot\delta\vec{r}$, where $\delta\vec{r}$ represents the displacement of the point where the force is being applied. Thus the virtual work in the present case is

$$2mg \times \frac{l}{4}\sin\left(\frac{\theta}{2}\right)\Delta\theta - F \times \frac{l}{2}\cos\left(\frac{\theta}{2}\right)\Delta\theta$$

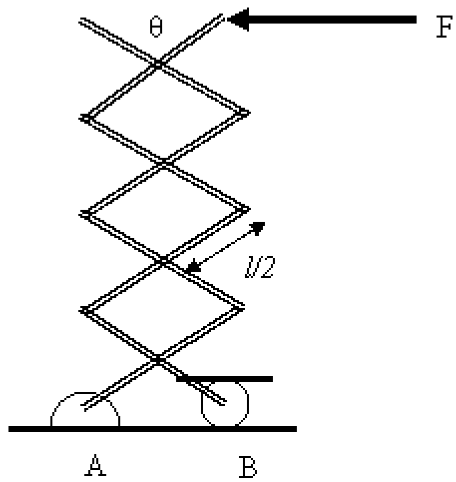
For equilibrium we equate this to zero to get

$$F = mg\tan\left(\frac{\theta}{2}\right)$$

which is the same result as obtained earlier.

So you see in both these examples that by applying the method of virtual work, we have bypassed calculating the constraint forces completely and that is what makes the method easy to implement in large systems. The way to learn the method well is to practice as many problems as possible. I will now solve some examples to demonstrate the usefulness of the method for large system. To start with let us take the example which we gave in the beginning - that of toy with made with bars.

Example 3: If there are *N* crossings in the folding toy shown in figure 9, what is the force required to keep the system in equilibrium?



A folding toy made up of 2N bars at N crossings in equilibrium under an applied force F. Length of each bar is l.

Figure 9

Again the degree of freedom = 1. The variable we use to describe the position of the mechanism is the angle between the bars i.e. θ. As the angle θ is changed to (θ+ Δθ), the upper end of the bar where force *F* is applied moves in the direction opposite to the force by

$$l\sin\left(\frac{\theta+\Delta\theta}{2}\right)-l\sin\left(\frac{\theta}{2}\right)=\frac{l}{2}\cos\left(\frac{\theta}{2}\right)\Delta\theta$$

Thus the virtual work done by *F* is

$$-F\times\frac{l}{2}\cos\left(\frac{\theta}{2}\right)\Delta\theta$$

On the other hand, the first crossing moves down by

$$\frac{l}{2}\cos\left(\frac{\theta}{2}\right)-\frac{l}{2}\cos\left(\frac{\theta+\Delta\theta}{2}\right)=\frac{l}{4}\sin\left(\frac{\theta}{2}\right)\Delta\theta$$

The second crossing by

$$\left\{l\cos\left(\frac{\theta}{2}\right)-l\cos\left(\frac{\theta+\Delta\theta}{2}\right)\right\}+\left\{\frac{l}{2}\cos\left(\frac{\theta}{2}\right)-\frac{l}{2}\cos\left(\frac{\theta+\Delta\theta}{2}\right)\right\}=\frac{3l}{4}\sin\left(\frac{\theta}{2}\right)\Delta\theta$$

and the *Nth* crossing moves down by

$$(N-1)\left\{l\cos\left(\frac{\theta}{2}\right)-l\cos\left(\frac{\theta+\Delta\theta}{2}\right)\right\}+\left\{\frac{l}{2}\cos\left(\frac{\theta}{2}\right)-\frac{l}{2}\cos\left(\frac{\theta+\Delta\theta}{2}\right)\right\}=\frac{l}{4}(2N-1)\sin\left(\frac{\theta}{2}\right)\Delta\theta$$

All these displacements are in the same direction as the force = *2mg* at each of the bar crossings. Thus the virtual work done by the weight of the mechanism is

$$2mg\times\frac{l}{4}\sin\left(\frac{\theta}{2}\right)\Delta\theta\times\sum_{n=1}^{N}(2n-1)=N^2\frac{mgl}{2}\sin\left(\frac{\theta}{2}\right)\Delta\theta$$

This gives a total virtual work done by the external forces to be

$$\left(N^2\frac{mgl}{2}\sin\left(\frac{\theta}{2}\right)\Delta\theta-F\times\frac{l}{2}\cos\left(\frac{\theta}{2}\right)\Delta\theta\right)$$

Equating this to zero for equilibrium gives

$$F=N^2 mg\tan\left(\frac{\theta}{2}\right)$$

For $N = 1$ the answer matches with that obtained in the case of only two bars in example 2 above. For larger N , the force required to keep equilibrium goes up by a factor of N^2 .

Example 4: A *6m* long electric pole of weight W starts falling to one side during rains. It is kept from falling by tying a strong rope at its centre of gravity (assumed to be right in the middle of the pole) and securing the other end of the rope on ground. All the relevant distances are given in figure 10. Assume that the lower end of the pole is like a pin joint. Under these conditions we want to find the tension in the rope using the method of virtual work.

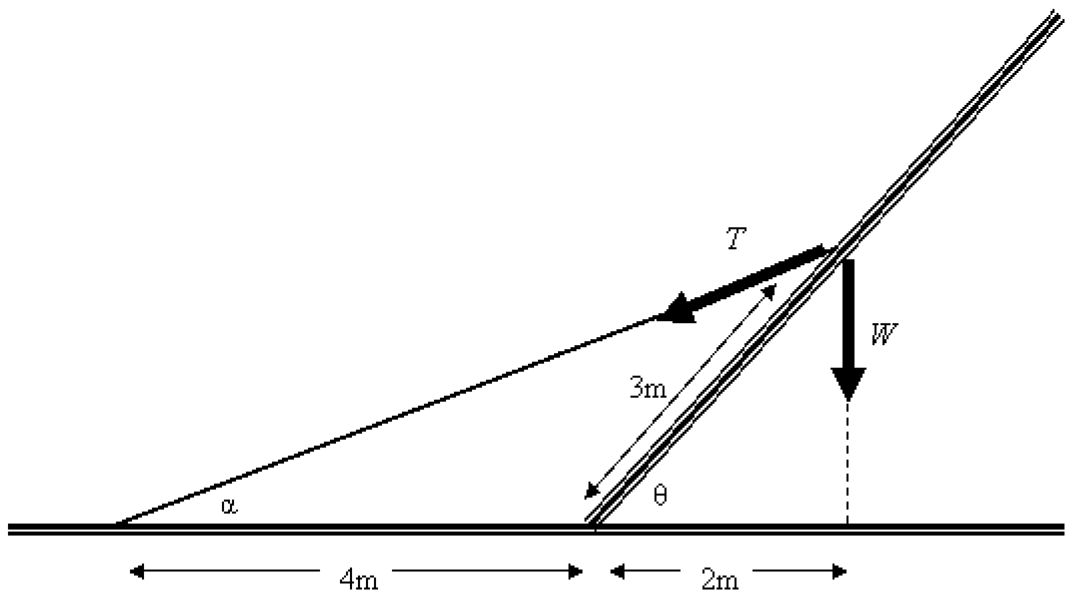


Figure 10

In this problem also there is only one degree of freedom θ. The constraint is that the pole can only rotate about the assumed pin joint at the ground. The constraint forces are the reactions at the ground and do no work on the pole when it rotates. There is also the constraint of the rigidity of the pole. Extend forces are *W* and *T*. By principle of virtual work when θ is changed to $(\theta + \Delta\theta)$, the total virtual work vanishes. If the centre of gravity moves up by Δy and to the left by Δx as θ is increased to $(\theta + \Delta\theta)$, the virtual work done is

$$\delta W = -(T\sin\alpha + W)\Delta y + T\cos\alpha\Delta x$$

which, when equated to zero, gives

$$T = \frac{W\Delta y}{\Delta x\cos\alpha - \Delta y\sin\alpha}$$

From the figure it is easy to see that

$$\sin\alpha = \sqrt{\frac{5}{41}} \quad \text{and} \quad \cos\alpha = \frac{6}{\sqrt{41}}$$

and (only the magnitude)

$$\Delta x = 3\sin\theta\Delta\theta = \sqrt{5}\Delta\theta \quad \text{and} \quad \Delta y = 3\cos\theta\Delta\theta = 2\Delta\theta$$

Substituting these in the expression for the tension gives

$$T = \frac{1}{2}\sqrt{\frac{41}{5}}W$$

This concludes the lecture on the method of virtual work. In the lecture, I have given you an introduction to the method assuming that constraints do no work. The method is really useful when there are many constraints and the system is complicated. It makes calculations easier by avoiding calculating constraint forces. The method also provides basis for simplifying dynamics calculations under constrained motion. You will be learning more about it in an advanced course.

Lecture 11
Motion in a plane: Introduction to polar coordinates

So far we have discussed equilibrium of bodies i.e. we have concentrated only on statics. From this lecture onwards we learn about the motion of particles and composite bodies and how it is affected by the forces applied on the system. Thus we are now starting study of dynamics.

When we describe the motion of a particle, we specify it by giving its position and velocity as a function of time. How the motion changes with time is given by the application of Newton's IInd Law. One such particle at position $\vec{r}$ moving with velocity $\vec{v}$ and acted upon by a force $\vec{F}$ is shown in figure 1. The force $\vec{F}$ gives rise to an acceleration $\vec{a} = \frac{\vec{F}}{m}$. Notice that in general the position, the velocity and the acceleration are not in the same direction.

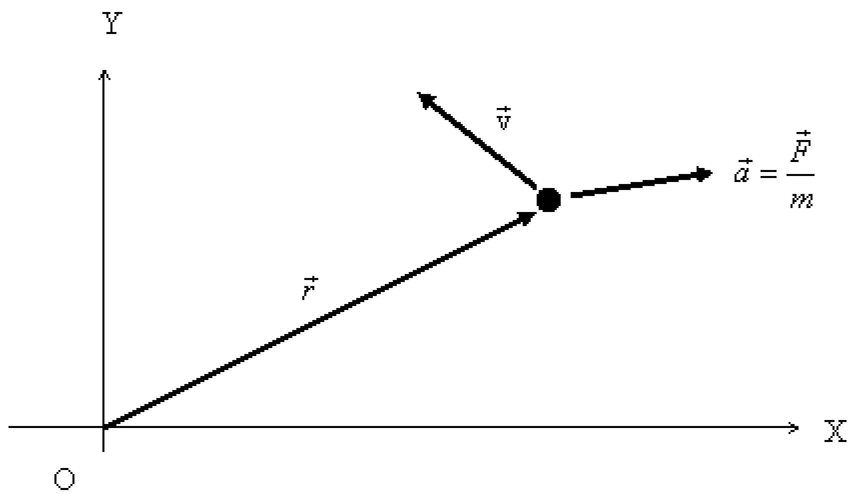


Figure 1

Each of these vectors is specified by giving its component along a set of conveniently chosen axes. For a particle moving in a plane, if we choose the Cartesian coordinate system (x-y axes) then the position is given by specifying the coordinates (x, y), velocity by its components $(\mathrm{v}_x, \mathrm{v}_y)$ and acceleration by its components $a_x = \frac{F_x}{m}$ and $a_y = \frac{F_y}{m}$. These are related by the relationship

$$\mathrm{v}_x = \frac{dx}{dt} \quad , \quad \mathrm{v}_y = \frac{dy}{dt}$$

and

$$a_x = \frac{d\mathrm{v}_x}{dt} = \frac{d^2x}{dt^2} \quad , \quad a_y = \frac{d\mathrm{v}_y}{dt} = \frac{d^2y}{dt^2}$$

These expressions are easily generalized to three dimensions by including the z-component of the motion also. However, in this lecture we will be focusing on motion in a plane only. With these components the equations of motion to be solved are

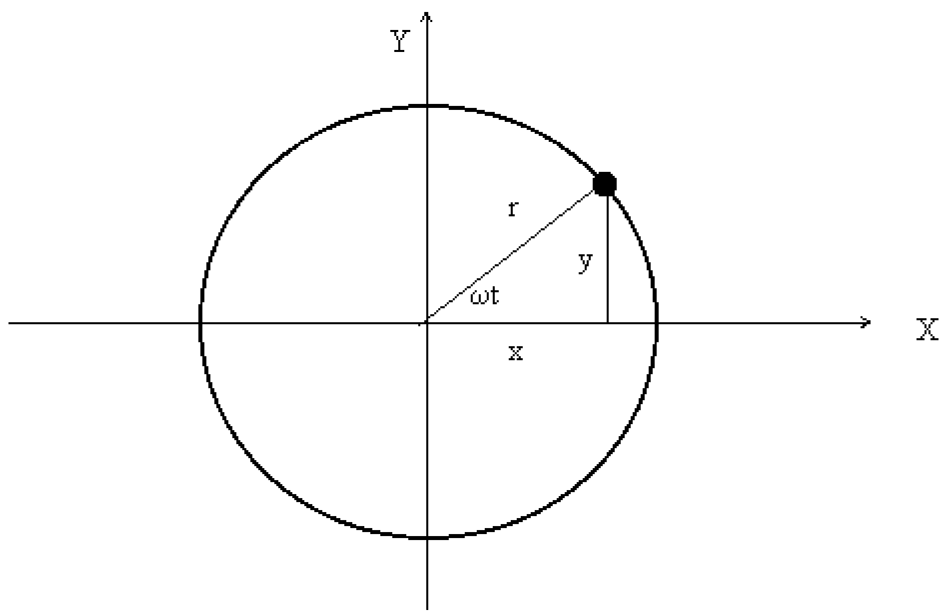
$$m\frac{d^2x}{dt^2} = F_x \quad , \quad m\frac{d^2y}{dt^2} = F_y$$

Coupled with the initial conditions solutions of these equations provide the velocity and position of a particle uniquely. However, the Cartesian system of coordinates is only one way of describing the motion of a particle. There arise many situations where describing the motion in some other coordinate system i.e., taking components along some other directions is move convenient. One such coordinate system is polar coordinates. In this lecture we discuss the use of this system to describe the motion of a particle. To introduce you to polar coordinates and how their use may make things easy, we start with the discussion of a particle in a circle.

Consider a particle is moving with a constant angular speed ω in a circle of radius R centered at the origin (see figure 2). Its x and y coordinates are given as

$$x = R\cos \omega t$$
$$y = R\sin \omega t$$

with both x and y being functions of time (see figure 2).



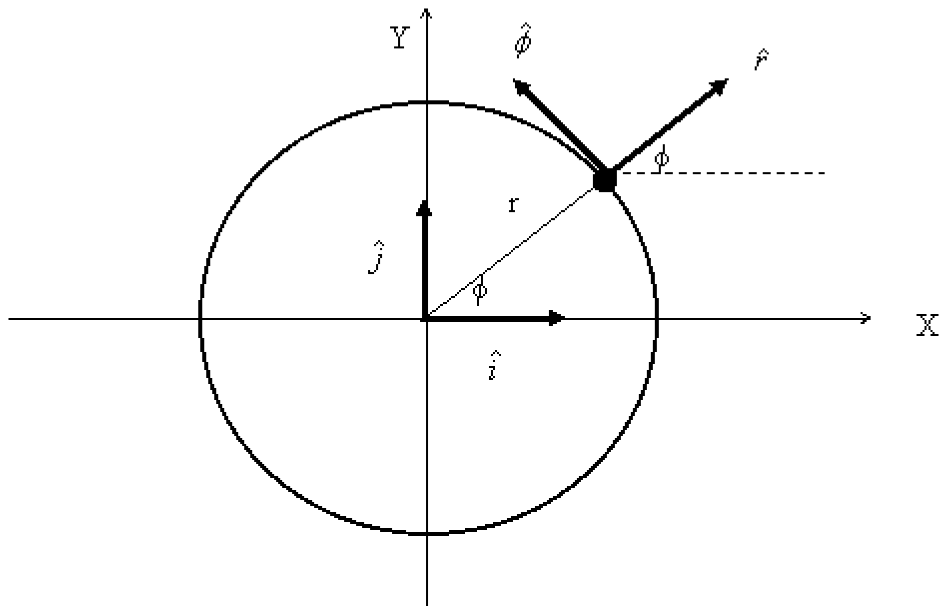
A particle moving with constant angular speed ω in a circle of radius R

Figure 2

On the other hand, if we choose to give the position of the particle by giving its distance r from the origin and the angle Φ that the line from the origin to the particle makes with x-axis in the counter-clockwise direction, then the position is given as

$$r = R$$
$$\phi = \omega t$$

In this coordinate system, r is a constant and Φ a linear function of time. Thus there is only one variable that varies with time whereas the other one remains constant. The motion description thus is simpler. These co-ordinates $(r \text{ and } \phi)$ are known as the planar polar coordinates. As expected, these coordinates are most useful in describing motion when there is some sort of a rotational motion. We will therefore find them useful, for example, in discussing motion of planets around the sun rotating bodies and motion of rotating objects.



Unit vectors $\hat{r}$ and $\hat{\phi}$ in polar coordinates

Figure 3

So to start with let us set up the unit vectors is polar co-ordinates (*r, Φ)* . Given a point (r, ϕ), the unit vector $\hat{r}$ is in outward radial direction and has magnitude of unity. The Φ unit vector is also of magnitude unity and is perpendicular to $\hat{r}$ and in the direction of increasing Φ (see figure 3). Obviously the dot product $\hat{r} \cdot \hat{\phi} = 0$. In term of the unit vectors in *x* and *y* direction these are given as

$$\hat{r} = \cos\phi\hat{i} + \sin\phi\hat{j}$$
$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

As is clear from these expression the direction of $\hat{r}$ and Φ is not fixed but depends on the angle Φ. On the other hand, it does not depend on r. If we go along a radius, $\hat{r}$ and Φ remain unchanged as we move (recall that two parallel vectors of same magnitude are equal). But that is not the case if Φ is changed.

The position a of a particle in polar co-ordinates to given by writing

$$\vec{r} = r\,\hat{r}$$

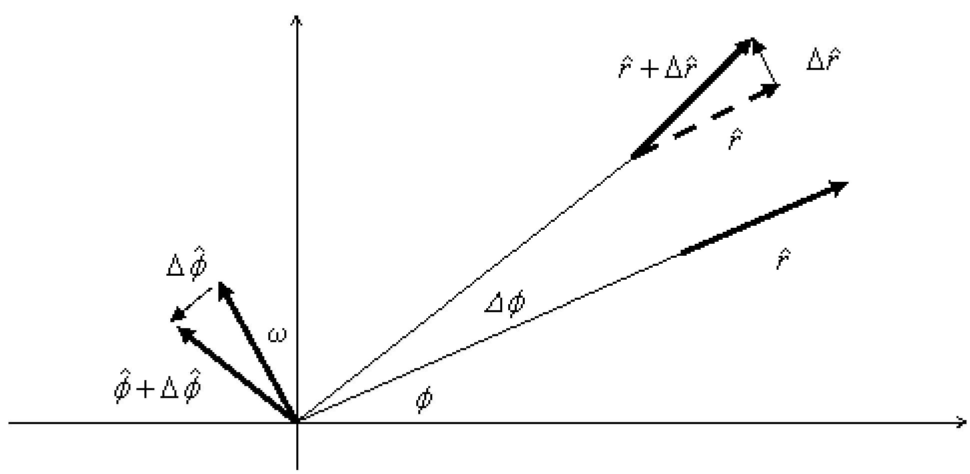
As particle moves about, $\vec{r}$ changes. Does the mean that the velocity

$$\vec{\mathrm{v}} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r}\ ?$$

The answer is no. As already discussed $\hat{r}$ is a function of *Φ*, the angle from the x-axis. Thus as a particle moves such that the angle Φ changes with time, the unit vector $\hat{r}$ also changes. Its derivative with respect to time is therefore not zero. Thus the correct expression for $\vec{\mathrm{v}}$ is

$$\vec{\mathrm{v}} = \left(\frac{dr}{dt}\right)\hat{r} + r\left(\frac{d\hat{r}}{dt}\right)$$

Let us now calculate $\left(\frac{d\hat{r}}{dt}\right)$. As already stated, $\hat{r}$ does not change as one moves radically in or out. Thus $\hat{r}$ changes only if Φ changes. Let us now calculate this change (figure 4)



Change in unit vector $\hat{r}$ and $\hat{\phi}$ as angle ϕ is changed to $\phi+\Delta\phi$

Figure 4

As is clear from the figure

$$\Delta\hat{r} = \Delta\phi\hat{\phi}$$
$$\Rightarrow \frac{d\hat{r}}{dt} = \left(\frac{\Delta\phi}{\Delta t}\right)\hat{\phi} = \dot{\phi}\hat{\phi}$$

where the dot on top of a quantity denotes its time derivative. The expression above can also be derived mathematically as follows:

$$\begin{aligned}\frac{d\hat{r}}{dt} &= \frac{d}{dt}\left(\cos\phi\hat{i} + \sin\phi\hat{j}\right)\\ &= \dot{\phi}\left(-\sin\phi\hat{i} + \cos\phi\hat{j}\right)\\ &= \dot{\phi}\hat{\phi}\end{aligned}$$

Thus the velocity of a particle is given as

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

We note that the unit vectors in polar coordinates keep changing as the particle moves because they are given by the particles current position. Thus even if a particle were moving with a constant velocity, the components of velocity along the radial and the directions will change. Let us calculate the velocity of a particle moving in a circle with a constant angular speed. For such a particle

$$\dot{r} = 0 \quad \text{and} \quad \dot{\phi} = \omega$$

so the velocity is given as

$$\vec{v} = R\omega\hat{\phi}$$

This is a well known result: the velocity of a particle moving in a circle with a constant angular speed is in the tangential direction and its magnitude is Rω. How about the acceleration in polar coordinates? This is the derivative of $\vec{v}$ with respect to time. Thus

$$\vec{a} = \left(\frac{d\vec{v}}{dt}\right) = \left(\frac{d}{dt}\right)(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$

As was the case with the unit vector $\hat{r}$, the unit vector $\hat{\phi}$ also is a function of the polar angle Φ and as such changes as the particle moves about. Thus in calculating the acceleration, time derivative of $\hat{\phi}$ also should be taken into account. From figure 4 it is clear that

$$\Delta\hat{\phi} = -\Delta\phi\hat{r}$$
$$\Rightarrow \frac{d\hat{\phi}}{dt} = -\left(\frac{\Delta\phi}{\Delta t}\right)\hat{r} = -\dot{\phi}\hat{r}$$

This can also be derived mathematically as

$$\begin{aligned}
\frac{d\hat{\phi}}{dt} &= \frac{d}{dt}\left(-\sin\phi\hat{i} + \cos\phi\hat{j}\right) \\
&= \dot{\phi}\left(-\cos\phi\hat{i} - \sin\phi\hat{j}\right) \\
&= -\dot{\phi}\hat{r}
\end{aligned}$$

Using this derivative and the chain rule for differentiation, we get

$$\begin{aligned}
\vec{a} &= \left(\frac{d}{dt}\right)(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) \\
&= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt} \\
&= \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}
\end{aligned}$$

You can see that the expression is a little complicated. The complexity of the expression arises because the unit vectors are changing as the particle moves. You can check for yourself that for a particle moving with a constant velocity, the expression above will give zero acceleration. Despite little complicated expressions for the acceleration, employing polar coordinates becomes really useful in situations where motion is circular-like as we will see in two standard examples later. Let us first go to one familiar example of a particle moving in a circle for which r = R , $\dot{\phi} = \omega$. This gives

$$\vec{a} = -R\omega^2\hat{r}$$

which is the correct answer for the centripetal acceleration. For this reason $r\dot{\phi}^2$ is known as the centripetal term. Let us now solve an example of mechanics using polar co-ordinates.

Example 1: A bead of mass *m* can slide without friction on a straight thin wire moving with constant angular speed 'ω' in a horizontal plane (figure 5). If we leave the bead with zero initial radial velocity at $r = R$, we wish to describe its subsequent motion and also find the horizontal force applied by the wire on the bead.

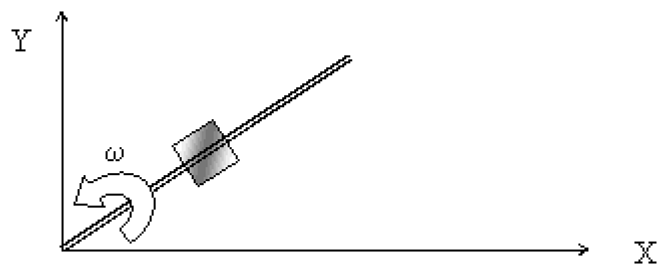


Figure 5

To see the usefulness of polar coordinates, try to write equations of motion for the bead in the Cartesian coordinates. This I leave for you to do. We solve the problem using polar co-ordinates. Thus at any instant the acceleration is given by the formula

$$\vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

We emphasize that the expression above gives the components of the acceleration along the radial and the *f* directions which are not fixed in space but are changing continuously. It is given that $\dot{\phi} = \omega$ (a constant) which also means that $\ddot{\phi} = 0$. The acceleration of the bead on the wire is therefore

$$\vec{a} = \left(\ddot{r} - r\omega^2\right)\hat{r} + 2\dot{r}\omega\hat{\phi}$$

Since there is no friction, the wire does not apply any radial force on the bead. Therefore

$$\ddot{r} - r\omega^2 = 0$$

You can check by substitution that the solution for the equation above is

$$r = Ae^{\omega t} + Be^{-\omega t}$$

where A and B are two constants to be determined from the initial conditions. Differentiating the equation above gives

$$\dot{r} = \omega\left(Ae^{\omega t} - Be^{-\omega t}\right)$$

Thus acceleration perpendicular to wire is

$$
\begin{aligned}
a_{\phi} &= 2\dot{r}\omega \\
&= 2\omega^2\left(Ae^{\omega t} - Be^{-\omega t}\right)
\end{aligned}
$$

So the horizontal force applied by wire is

$$
F_{\phi} = ma_{\phi} = 2m\omega^2\left(Ae^{\omega t} - Be^{-\omega t}\right)
$$

Of course because the unit vectors employed change direction continuously, the force above is also in different directions at different times with the magnitude given by the expression above. To determinate A and B, we substitute t = 0 in the expressions derived for the radius and the radial speed and equate them to their vales given at that time. This gives

$$
\begin{aligned}
&A + B = R \\
&A - B = 0 \\
&\Rightarrow A = B = \frac{R}{2}
\end{aligned}
$$

This leads to the answer

$$
\begin{aligned}
r &= \frac{R}{2}(e^{\omega t} - e^{-\omega t}) \\
\dot{r} &= \frac{R\omega}{2}(e^{\omega t} - e^{-\omega t}) \\
F &= m\omega^2 R(e^{\omega t} - e^{-\omega t})
\end{aligned}
$$

Example 2: A particle, tied to a string, is moving on a smooth frictionless table in a circle of radius r_0 with an angular speed ω_0. The string is pulled in slowly through a hole in the middle of the table with constant speed *V*. We want to find the change in its speed as a function of time and also the force required for the string to be pulled (figure 6).

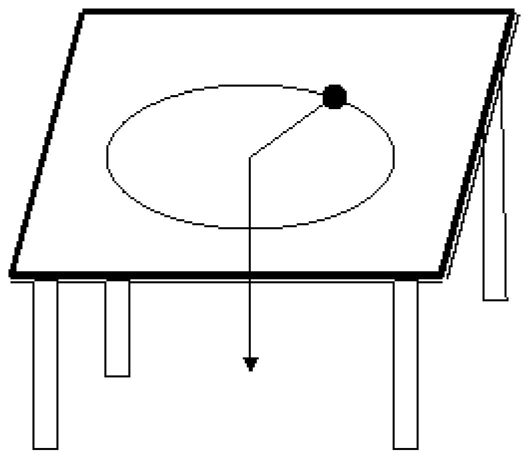


Figure 6

The mass, when pulled in, is moving under the influence of an inwardly directed radial force $-F\hat{r}$. Although the force keeps changing its direction depending upon where the particle is, it always remains radial. The expression for the acceleration of the particle in the polar coordinates is

$$\vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

Since it is given $\dot{r} = -V$, which means $\ddot{r} = 0$, and the force is only in $-\hat{r}$ direction, we have

$$\ddot{r} - r\dot{\phi}^2 = -r\dot{\phi}^2 = -\frac{F}{m}$$
$$\text{or} \quad mr\dot{\phi}^2 = F$$

Since there is no force component in the Φ direction, we have

$$r\dot{\phi}^2 + 2\dot{r}\dot{\phi} = 0$$

Multiply both sides of this equation by *r* to get

$$r^2\dot{\phi}^2 + 2r\dot{r}\dot{\phi} = 0$$
$$\text{or} \quad \frac{d}{dt}\left(r^2\dot{\phi}\right) = 0$$
$$\text{or} \quad r^2\dot{\phi} = \text{constant}$$

Since $r = r_0 - Vt$, the equation above gives

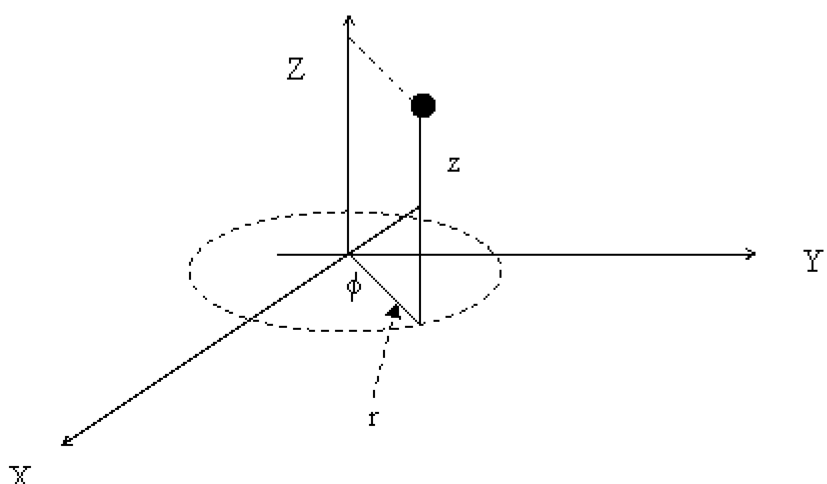
$$r_o{}^2\omega_o = (r_o - Vt)^2\,\omega$$
$$\text{or}\quad \omega = r_o{}^2\omega_o/(r_o - Vt)^2$$

The force $F = mr\dot{\phi}^2$ pulling the string in is therefore

$$F = m\frac{r_o{}^4\omega_o}{(r_o - Vt)^3}$$

In solving this example, we see that for forces in radial direction $r^2\omega = \text{constant}$, which is nothing by a statement of the conservation of angular momentum. We will discuss it more later when we study angular momentum.

After introducing the planar polar coordinates, we nor briefly describe what are the other coordinate systems in three dimensions. A natural extension of planar coordinates in the cylindrical coordinate system. This arises when we add the third-z direction to planar polar coordinates. See figure 7.



Cylindrical coordinates of a particle

Figure 7

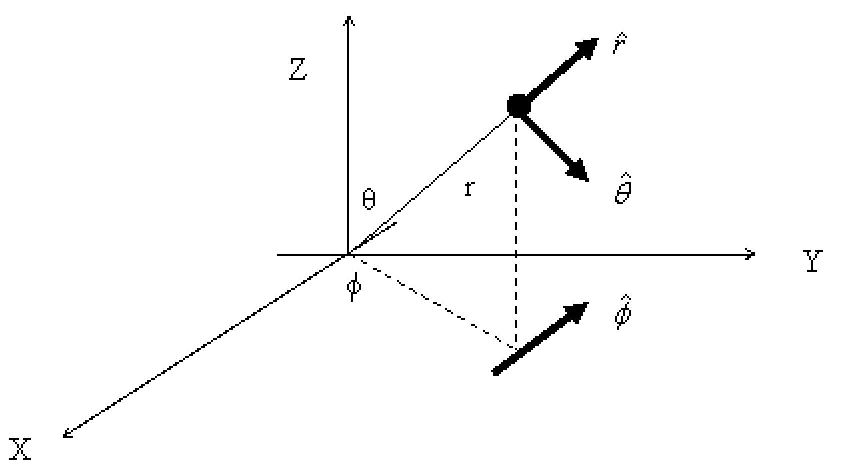
The position of a particle is described by (r, ϕ, z) with the corresponding unit vectors being $(\hat{r}, \hat{\phi}\,\text{and}\,\hat{z})$. In this case the $\hat{z}$ unit vector is a constant and $(\hat{r}, \hat{\phi})$ are given as in the planar polar co-ordinates so that

$$\hat{r} = \cos\phi\,\hat{i} + \sin\,\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\,\phi\,\hat{i} + \cos\phi\,\hat{j}$$
$$\hat{z} = \hat{k}$$

Thus the expressions for all the quantities are similar to those for planar polar co-ordinates except that $\hat{z}$ direction is also added. As a result,

$$\vec{r} = r\hat{r} + z\hat{z}$$
$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$
$$\vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi} + z\hat{z}$$

We now introduce another set of coordinates, the spherical polar coordinates, in three dimensions. A point in these coordinates is specifically by its distance from the centre r , the angle θ that the line joining the point to origin makes with the z-axis and the angle Φ that the projection of this line on the (xy) plane makes with the x-axis. Thus a point is specified by (r, θ, ϕ) (see figure 8).



Spherical coordinates of a point and the unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$

Figure 8

Thus (x, y, z) co-ordinates for a point (r, θ, ϕ) are

$$x = r\sin\,\theta\cos\phi$$
$$y = r\sin\,\theta\sin\,\phi$$
$$z = r\cos\theta$$

The unit vectors are given as $\hat{r}, \hat{\theta}\, and\, \hat{\phi}$ with

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$\hat{\theta}$ unit vector points in a direction below the (xy) plane making an angle θ from the (xy) plane. So it is given as

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

And $\hat{\phi}$ is in the (xy) plane and is given as

$$\phi = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

which is the same as for planar polar coordinates. As is clear, the unit vectors in this case are also position dependent and change as the particle position changes. This affects the expression for velocities and acceleration when they are expressed in spherical coordinates.

Let us evaluate the time derivatives of $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ geometrically. The unit vector $\hat{r}$ does not depend on r but changes with θ . This gives

$$\Delta\hat{r}(\theta \to \theta + \Delta\theta) = \Delta\theta\hat{\theta}$$

Similarly when θ is fixed and Φ changes, we get

$$\Delta\hat{r} = (\phi \to \phi + \Delta\phi) = \sin\theta\Delta\phi\hat{\phi}$$

When we combine the two results we get

$$\Delta\hat{r} = \Delta\theta\hat{\theta} + \sin\theta\Delta\phi\hat{\phi}$$

which gives

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}$$

Thus the expression for velocity in spherical coordinates is

$$\begin{aligned}\vec{\mathrm{v}} = \dot{\vec{r}} &= \frac{d}{dt}(r\hat{r}) \\ &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}\end{aligned}$$

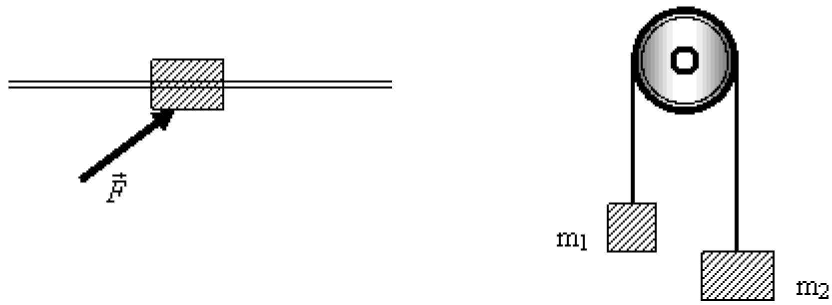
We leave the calculation of $\dot{\hat{\theta}}$ and $\dot{\hat{\phi}}$ and the acceleration as an exercise. We end this brief introduction to spherical coordinates by noting that spherical polar coordinates can be those of as two plane polar coordinates systems : one the plane of radius vector and the z-axis with (r, θ) as planar coordinates and the other the (xy) plane with $(r \sin \theta, \phi)$ as the planar polar coordinates.

Lecture 12
Motion with constraints

In this lecture we are going to deal with motion of particles when they move under constraints applied on their motion. Of course the motion is determined by Newton 's second law i.e., by solving the equation of motion

$$m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

where $\vec{F}$ is the total force – which is the sum of the externally applied and those arising from other particles as well as the constraints in the system - acting on a body of mass *m* and is producing an acceleration $\vec{a} = \frac{d^2\vec{r}}{dt^2}$. Recall from *lecture 9* that constraints are the restrictions applied on the movement of a body by various means and are brought about by *constraint forces* . For example, I may restrict the body to move along a straight wire (see figure 1). In that case the component of $\vec{F}$ only along the wire will affect the motion of the mass (if there is no friction) and its perpendicular component will be nullified by the normal reaction of the wire, which is the constraint force in this case. As another common example of constrained motion take the motion of two masses at the end of a rope going over a frictionless pulley (Atwood's machine) also shown in figure 1.



Two examples of constrained motion

Figure 1

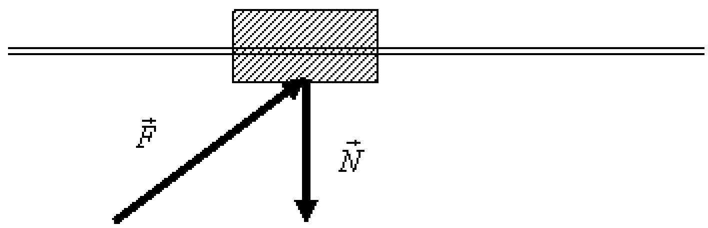
In this case also, the motion of one mass is determined by not only by the gravitational force on it alone but also by the weight of the other mass. Thus the two masses are not fully free to move under their own weight and the motion is constrained. The constrained is brought about through tension in the rope, which is then the constraint force.

We have seen two simple examples of constrained motion. We make an observation that constraints can be caused either by restricting the motion externally, as was the case for a mass on a wire, or by the presence of other bodies that are themselves moving, as in the example of two masses over a pulley. In lecture 9 we had introduced these concepts and stopped at that. However, for obtaining the positions and velocities of particles under constraints, we wish to express these constraints mathematically and account for them while solving the equations of motion. This is what this lecture is going to be about.

Let us start with the example of a mass on a straight wire (say in x direction). The constraint that the mass moves only in the x-direction is equivalent to saying that

$$y = \text{constant}$$
$$z = \text{constant}$$

This is how we mathematically express the constraint that the mass moves only along the x-axis. As pointed out earlier, to keep the y and the z coordinates of the mass unchanged, the wire applied a normal force on the mass to cancel the perpendicular (to the wire) component of the applied force so that the net force is along the wire. This normal reaction is the constraint force (figure 2). Notice that all that the wire does to the mass, as far as its motion is concerned, is represented by this force.

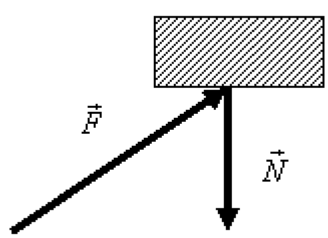


A mass constrained to move on a wire under applied force $\vec{F}$. Normal force $\vec{N}$ is the constraint force. Note that the sum of these two forces is along the wire.

Figure 2

.

To study the motion of the mass all I need to look at are only the forces – external and constraint forces - acting on the mass. In this case the wire is represented by the normal force that it applies. Recall from *lecture 4* that such a diagram is called a *free-body diagram* . The advantage of drawing a free-body diagram is that it identifies the relevant quantities to write the equation of motion. In the present case the free-body diagram of the mass is given in figure 3.



Free-body diagram of a mass on a wire

Figure 3

Let us now write the equations of motion for the body in terms of its x, y and z -components :

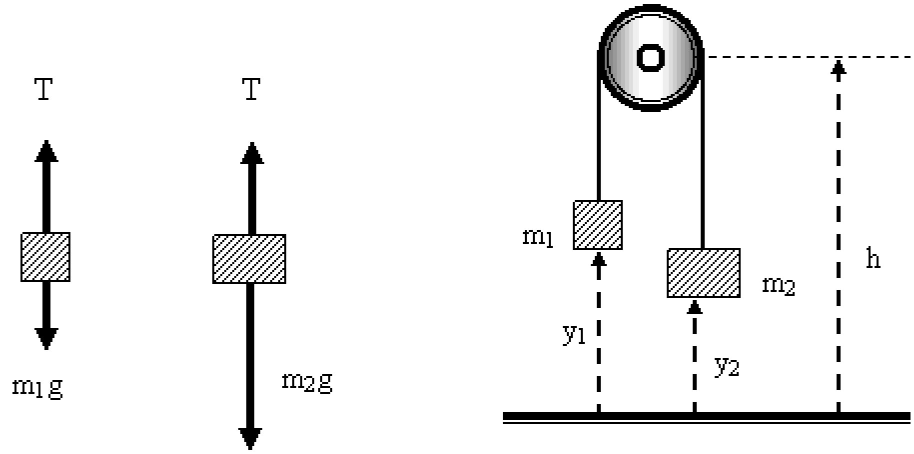
$$m\ddot{x} = F_x$$
$$m\ddot{y} = F_y + N_y$$
$$m\ddot{z} = F_z + N_z$$

Let us count how many unknown are there? The unknowns are x , y , z , N_y , and N_z , numbering five ($\vec{F}$ is given). But there are only three equations. How do we find the other two equations? For this recall that the two of the unknowns, N_x and N_y , arise because of the constraints. And it is these constraints that provide the two more equations needed for a solution. The constraints that y = constant and z = constant imply that

$$\ddot{y} = 0$$
$$\ddot{z} = 0$$

With these two additional equations, we now have five equations and five unknowns. Thus and we can solve for x , y , z and N_x and N_y in terms of given parameters of the problem.

Let us now look at the other problem of two masses hanging on the sides of a frictionless pulley (see figure 1), a special case of Atwood's Machine. For simplicity we take the pulley and the rope to be massless. Let the masses be m_1 & m_2 . In this problem also the motion is in only one direction i.e. the vertical direction so we are going to ignore the other two dimensions. In this problem the constraint is that the two masses move together and it is effected by the rope. As noted above, the force of constraint therefore is the tension T in the rope. Let us now make their free-body diagrams for the two moving masses m_1 and m_2. We measure all distances from the ground and let the distance of m_1 be y_1 and that of m_2 is y_2 . Please see figure 4.



Free-body diagrams of m_1 and m_2 and their distances y_1 and y_2 from the ground. The pulley is at height h.

Figure 4

Equation of motion for m_1 and m_2 are

$$m_1\ddot{y}_1 = T - m_1 g$$
$$m_2\ddot{y}_2 = T - m_2 g$$

The tension T is the same on both sides because rope and pulley both are massless and the pulley is also frictionless. These are two equations and there are three unknowns: y_1 , y_2 and T . The tension T arises because of constraint so the constraint itself provides the desired third equation. In this case the constraint is that the length of the rope is constant. This can be expressed mathematically as (see figure 4 for meaning of symbols)

$$(h - y_1) + (h - y_2) = \text{length of rope} - \pi R = \text{constant}$$

where R is the radius of the pulley. Differentiating this equation twice with respect to time gives

$$\ddot{y}_1 + \ddot{y}_2 = 0 \quad \text{or} \quad \ddot{y}_1 = -\ddot{y}_2$$

We now have three equations for three unknowns:

$$m_1\ddot{y}_1 = T - m_1 g$$
$$m_2\ddot{y}_2 = T - m_2 g$$
$$\ddot{y}_1 = -\ddot{y}_2$$

Solving these equations gives

$$\ddot{y}_1 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g \qquad \text{and} \qquad \ddot{y}_2 = -\left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$$

a result that you already know. Thus if $m_2 > m_1$, m_1 accelerates up.

Through these two simple examples, I have identified sequential steps that we take in solving a problem involving constraints I now summarize these steps:

1. Identify the constraints and forces of constraints in the given problem;
2. Make free body diagrams of different bodies taking part in the motion. Let me remind you in making free body diagram take the body and show all the forces - applied and those of constraints - on the body;
3. Write equations of motion for each subsystem/body. At this stage the number of equations will be less than the number of variables in the problem;
4. Write the constraint equations. They will provide the missing equations (This happens because each constraint introduces a constraint force which becomes the additional unknown);
5. Solve the equations.

Let us us now apply the procedure outlined above to slightly more difficult examples.

Example 1: There are three massless and frictionless pulleys P1, P2 and P3. P1 and P2 are fixed and P3 can move up and down, as shown in figure 5. A massless rope R1 passes over the pulleys as shown and two masses m_1 and m_2 attached at its ends. A third mass m_3 is hanging from P3 by a rope R2 of fixed length. Find the acceleration of the three masses.

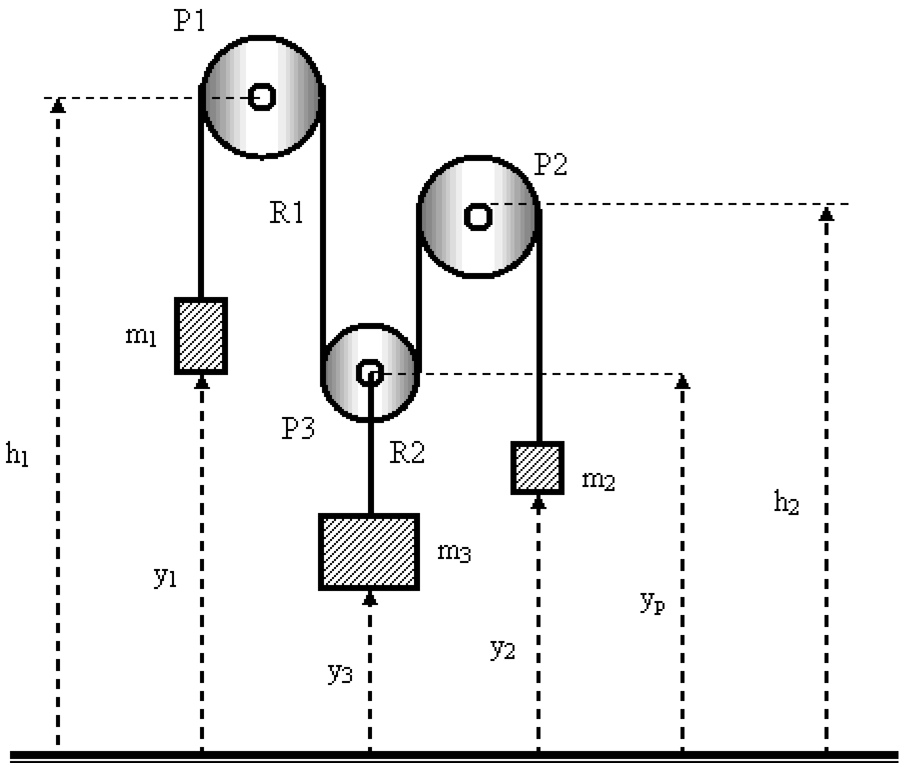
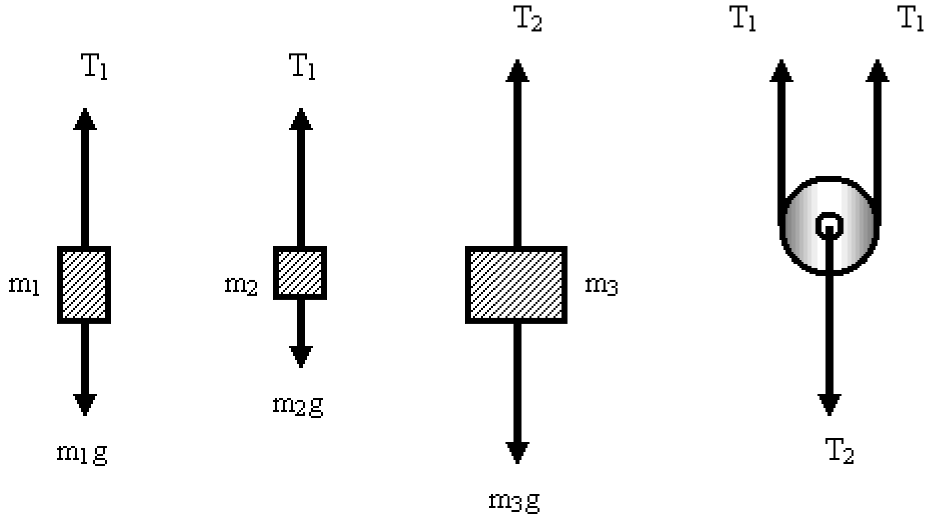


Figure 5

In figure 5 we have also shown the distances of different pulleys and masses from the ground, with the vertically up direction taken to be positive. The heights h_1 and h_2 of pulleys P1 and P2, respectively, are fixed whereas height y_p of pulley P3 can change. We go about solving the problem according to the steps given above.

Step 1: We identify two constraints and the forces of constraints as: rope R1 has fixed length with the force of constraint being tension T_1 in the rope. The other constraint is that rope R2 has fixed length with the tension T_2 in the rope as the constraint force. Because of massless pulleys and ropes and frictionless surfaces T_1 is the same throughout rope R1.

Step 2 : Make free-body diagrams of the subsystems. We consider only those subsystems that can move. Thus we make free-body diagram of each mass and the pulley P3 as shown in figure 6.



Free-body diagram for the three masses and pulley P3

Figure 6

Step 3 : By looking at the free-body diagrams, write equations of motion for each subsystem. In terms of the distances shown in figure 5, we get

$$m_1\ddot{y}_1 = T_1 - m_1 g$$
$$m_2\ddot{y}_2 = T_1 - m_2 g$$
$$m_3\ddot{y}_3 = T_2 - m_3 g$$

and because the pulley is massless

$$T_2 - 2T_1 = 0$$

Thus equations of motion give four equations. However there are six unknowns viz. y_1, y_2, y_3, T_1, T_2, and y_p. Their number exceeds the number of equations obtained so far by two.

Step 4 : The additional two equations are provided by the constraint equations. The constraint that rope R1 is of fixed length is expressed as (see figure 5 for the variables used)

$$(h_1 - y_1) + (h_1 + h_2 - 2y_p) + (h_2 - y_2) = \text{constant}$$
$$\text{or} \quad y_1 + y_2 + 2y_p = \text{constant}$$

Differentiating this equation twice with respect to time gives

$$\ddot{y}_1 + \ddot{y}_2 + 2\ddot{y}_p = 0$$

The second constraint that rope R2 is of fixed is equivalent to

$$y_p - y_3 = \text{constant}$$

which upon differentiating gives

$$\ddot{y}_p - \ddot{y}_3 = 0$$

Thus the equations that describe the motion of the system fully are:

$$\begin{aligned} m_1\ddot{y}_1 &= T_1 - m_1 g & \qquad T_2 - 2T_1 &= 0 \\ m_2\ddot{y}_2 &= T_1 - m_2 g & \qquad \ddot{y}_1 + \ddot{y}_2 + 2\ddot{y}_p &= 0 \\ m_3\ddot{y}_3 &= T_2 - m_3 g & \qquad \ddot{y}_p - \ddot{y}_3 &= 0 \end{aligned}$$

I will leave Step 5 – that is solving the equations - for you to do but give you partial answer. It is

$$\ddot{y}_3 = \frac{4m_1m_2 - (m_1 + m_2)m_3}{4m_1m_2 + (m_1 + m_2)m_3} g$$

I would now like you to try a similar problem but with slight difference. Let us attach the centre of the third pulley to a spring of spring constant *k* (see figure 7). Then find the equations of motion for the two masses and solve them.

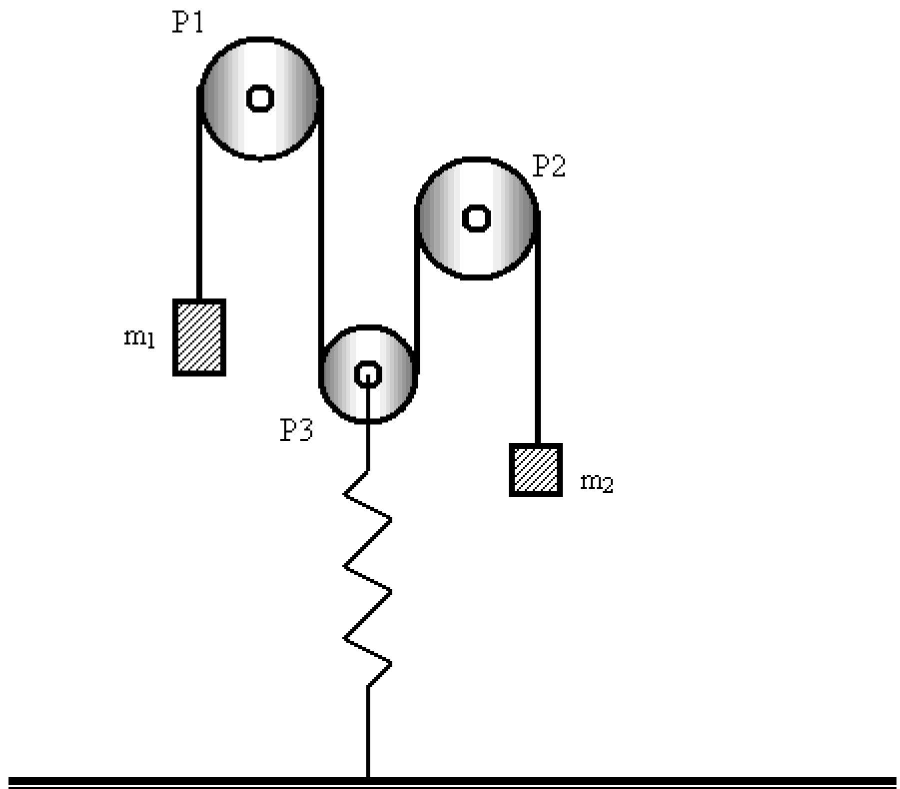


Figure 7

Example 2 : As another example of constrained motion we take a small block of mass *m* sliding down on a cylindrical surface from its top (figure 8). The question we ask is at what angle from the horizontal would the mass slide off the surface of the cylinder.

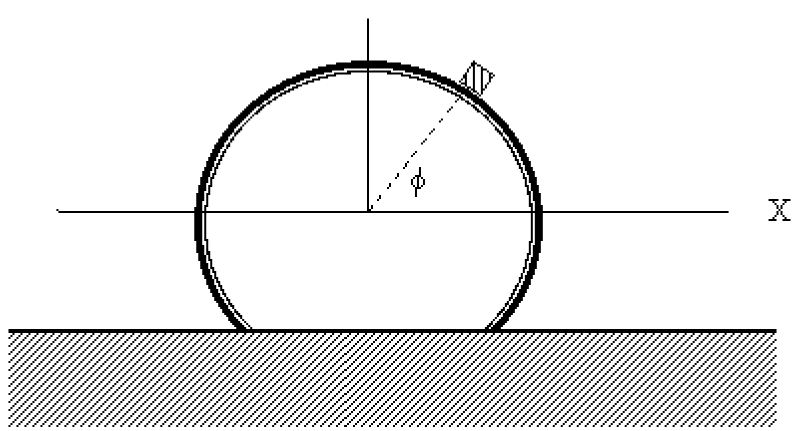
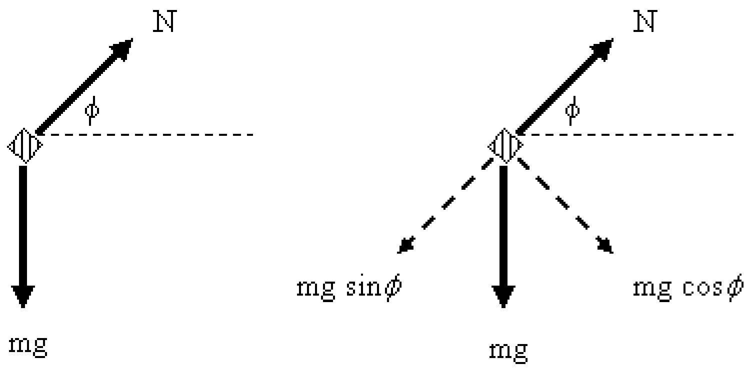


Figure 8

Since this problem involves motion along a circular path I would use planar polar coordinates. I take the origin at the centre of the cylinder and let the x-axis be along the horizontal and y-axis along the vertical. Assume that the radius of the cylinder is R . The constraint in this problem is that r = *constant* = R. The corresponding constraint force is the normal reaction N of the cylindrical surface on the block. The free-body diagram of the mass on the cylinder is shown in figure 9.



Free-body diagram of a mass on a cylinder (left) and the components of the weight mg in the radial and the ϕ directions (right)

Figure 9

We now write the equations of motion in the planar polar coordinates. That gives in the $\hat{r}$ direction

$$m(\ddot{r} - r\dot{\phi}^2) = N - mg\sin\phi$$

and in the $\hat{\phi}$ direction

$$m(r\ddot{\phi} - 2\dot{r}\dot{\phi}) = -mg\cos\phi$$

We again have three variables $(r, \phi \text{ and } N)$ but only two equations. The third equation is provided by the equation of constraint i.e.

r = *constant* = R

which gives

$$\dot{r} = \ddot{r} = 0$$

With this the equations to be solved are

$$-mR\dot{\phi}^2 = N - mg\sin\phi \qquad mR\ddot{\phi} = -mg\cos\phi$$

To solve these we use

$$\ddot{\phi} = \frac{d}{dt}(\dot{\phi}) = \frac{d}{d\phi}(\dot{\phi})\cdot\frac{d\phi}{dt}$$
$$= \dot{\phi}\cdot\frac{d}{d\phi}(\dot{\phi})$$
$$= \frac{1}{2}\frac{d}{d\phi}(\dot{\phi}^2)$$

Substituting this in the equation for $\ddot{\phi}$ above gives

$$\frac{mR}{2}\frac{d}{d\phi}(\dot{\phi}^2) = -mg\cos\phi$$
$$\Rightarrow R\dot{\phi}^2 = -2g\sin\Big|_{\pi/2}^{\phi}$$
$$= 2g(1-\sin\phi)$$

This when substituted in the equation for $R\dot{\phi}^2$ leads to

$$3mg\sin\phi = N + 2mg$$

The point when the mass slips off the cylinder is where *N* becomes zero. So the corresponding ϕ is given by

$$3mg\sin\phi = 2mg \quad \text{or} \quad \sin\phi = \frac{2}{3}$$

Example 3: Let us take one more example of constrained motion when two bodies are involved. I put a block of mass *m* on a wedge of mass M with wedge angle θ (see figure 10). The wedge is free to move on a frictionless plane. There is no friction also between *m* and *M* . We wish to find the resulting motion.

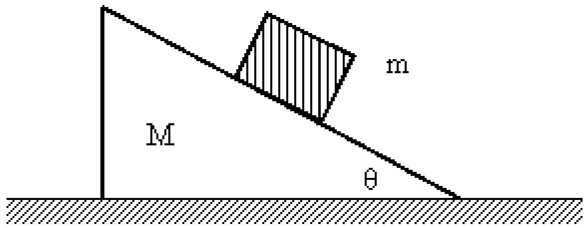


Figure 10

There are clearly two subsystems, the masses *m* and *M* . There are two constraints in the system. Constraint one is that the mass *m* moves along the edge of the wedge so its *x* and *y* components are not independent. The other constraint is that the wedge moves only in the x direction. The constraint forces are obviously the normal reaction N_1 on mass *m* by the wedge and the normal reaction N_2 on the wedge by the ground. The free-body diagrams for the two subsystems are as shown in figure 11.

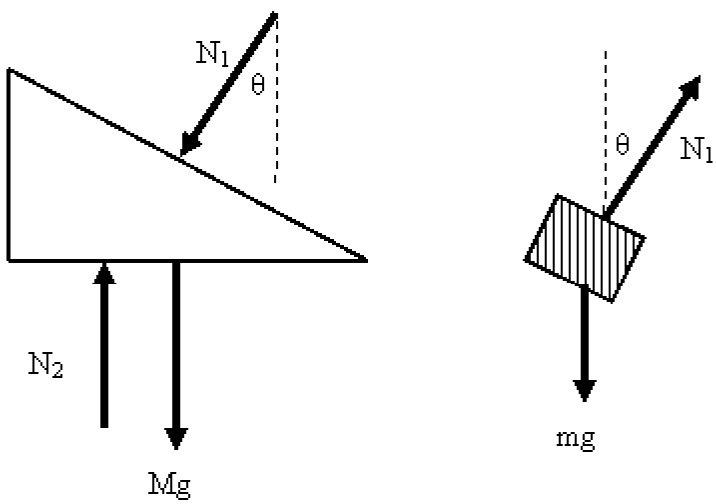


Figure 11

Notice that in the free-body diagram of the wedge, there is no *mg* of block. It is all accounted for by N_1. To set up the equations of motion, let us choose our co-ordinates system a follows (see figure 12): Let the coordinate of the right-hand side lower corner of the wedge be given the co-ordinates (x_1 y_1) and let the co-ordinates of the block be (x_2 y_2).

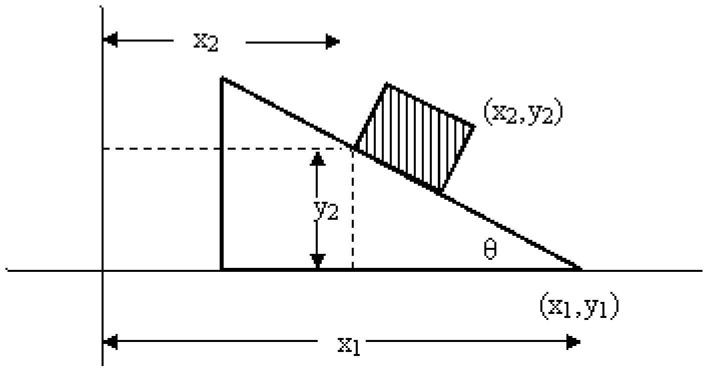


Figure 12

The equations of motion in terms of these coordinates are:

$$m\ddot{x}_2 = N_1 \sin\theta \qquad M\ddot{x}_1 = -N_1 \sin\theta$$
$$m\ddot{y}_2 = N_1 \cos\theta - mg \qquad M\ddot{y}_1 = -N_1 \cos\theta - mg + N_2$$

For the six variables - $(x_1 y_1), (x_2 y_2), N_1$ and N_2 - of the system, we need two more equations, which are provided by the constraints equations. These are

$$y_1 = 0 \text{ which gives } \dot{y}_1 = \ddot{y}_2 = 0$$

and

$$\frac{y_2}{x_1 - x_2} = \text{constant} = \tan\theta$$

which gives

$$\dot{y}_2 = (\dot{x}_1 - \dot{x}_2)\tan\theta \quad \text{and} \quad \ddot{y}_2 = (\ddot{x}_1 - \ddot{x}_2)\tan\theta$$

Thus the equations to be solved are

$$m\ddot{x}_2 = N_1 \sin\theta \qquad M\ddot{y}_1 = -N_1 \cos\theta - mg + N_2$$
$$m\ddot{y}_2 = N_1 \cos\theta - mg \qquad \ddot{y}_1 = 0$$
$$M\ddot{x}_1 = -N_1 \sin\theta \qquad \ddot{y}_2 = (\ddot{x}_1 - \ddot{x}_2)\tan\theta$$

These equations can now be solved to get all the variables as a function of time. That task is left for you. I'll leave you with answers for N_1 :

Motion with friction and drag

We have been looking at the constrained motion of particles and found that in solving the problems we make free-body diagrams and look at the motion of each subsystem independently. Then the motion of individual subsystem is linked through constraints that they impose on each other. The example that we took were Atwood's machine and a mass sliding on a wedge. However, in these examples we neglected a ubiquitous force which is the force of friction. In this lecture we take this into account and solve problems involving the friction

We would take into account two kinds of frictional forces - one that arises when two solid bodies are in contact and the other that arises when a body is moving through a liquid, the viscous force. Let us first consider the case when two solid bodies are moving against each other. A detailed discussion about the nature of frictional force and its relationship with the normal reaction has already been presented in lecture 6. We start with a review of the main points discussed there.

If there is a tendency between two bodies to slide against each other, or if one body is sliding over a surface, the friction between the two bodies resists this motion. Question is whether this is a constant force or adjusts itself. It is experimentally observed that the maximum frictional force $f_{\max}$ that a surface can apply on an object is

$$f_{\max} = \mu N$$

where N is the normal reaction of the surface on the body and μ is the coefficient of friction; its value is different for the static and dynamic case. Thus there are two coefficients of friction between two surfaces: static coefficient of static friction μ_s and the coefficient of dynamic friction μ_k, with the latter being smaller than the former. Further, μ_s is always observed to be less than 1. And the direction of frictional force is such that it opposes the motion or the tendency to move

Let us now take a couple of standard examples involving friction similar to those solved in lecture 6.

Example 1: We put a block of 5kg on top a 10kg block. They are then attached through a massless and frictionless pulley to a mass M as shown in figure 1. The coefficient of friction between all surfaces for both static and dynamic friction is 0.5. What is the acceleration for (a) $M = 20kg$ and (b) $M = 40kg$ ($g = 9.8m/s^2$)?

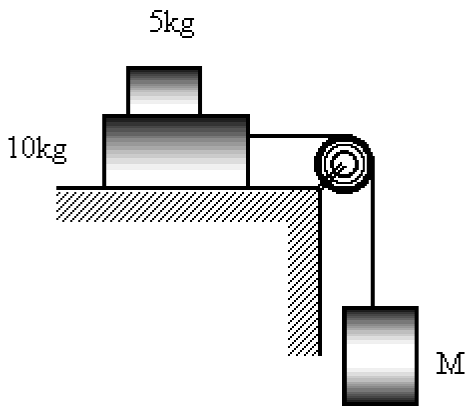
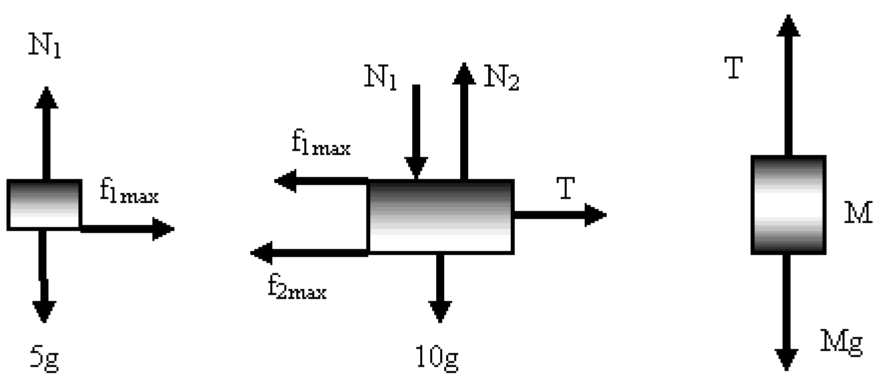


Figure 1

What we should see in solution of this problem is the maximum possible acceleration that the *5kg* block can have, and then solve for the mass M_0 that will give this acceleration for both the *5kg* and the *10kg* blocks. If M is less than M_0, both the blocks will move together. On the other hand, if M exceeds M_0, the blocks will slip on each other.

To start the calculations I show in figure 2 the free body diagrams of all the masses with maximum possible friction



Free body diagrams of all the blocks in figure with maximum possible friction.

Figure 2

Looking at the 5kg block, we see that

$$N_1 = 5g \qquad \Rightarrow \qquad f_{1\max} = .5 \times 5 \times 9.8 = 24.5N$$

The maximum possible acceleration for the 5kg mass is

$$a_{1\max} = \mu g = 4.9ms^{-2}$$

Let me now calculate M_0 corresponding to this acceleration. The corresponding equations for the 10kg block are

$$N_2 = N_1 + 10g = 147N \qquad \Rightarrow \qquad f_{2\max} = 0.5 \times 147 = 73.5N$$
$$T - f_{1\max} - f_{2\max} = 10a_{1\max} \qquad \Rightarrow \qquad T = 147N$$

The equation of motion for the mass *M* then gives M_0 as follows

$$M_0 g - T = M_0 a_{1\max} \qquad \Rightarrow \qquad M_0 = \frac{T}{g - a_{1\max}} = 30kg$$

Now I answer the question asked in the problem.

(a) For 20 kg mass, let the friction between the blocks be f. Then we have

$$f = 5a \qquad T - f - f_{2\max} = 10a \quad \text{and} \quad 20g - T = 20a$$

These equations lead to the acceleration of the system as follows

$$20g - f_{2\max} = 35a \qquad \Rightarrow \qquad a = 3.5ms^{-2}$$

(b) M = 40kg. Although I have already shown you that in this case the two blocks will slide on each other. Let me show this to you again in another way. Assume that the blocks move together. In that case the acceleration of the assembly will be

$$a = \frac{40g}{55} = 7.12ms^{-2}$$

But this is larger than the maximum possible acceleration for the 5kg block, so the assembly cannot move together. Under these conditions the equations for the 10kg block and the mass M are

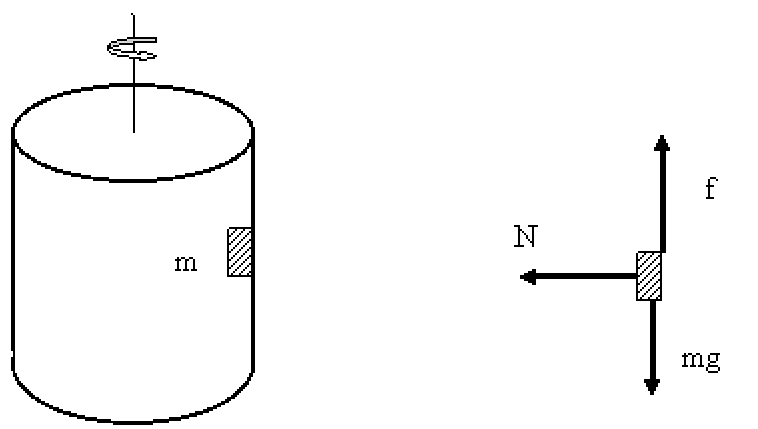
$$T - f_{1\max} - f_{2\max} = 10a \quad \text{and} \quad 40g - T = 40a$$

which gives

$$a = \frac{40g - f_{1\max} - f_{2\max}}{50} = 5.88ms^{-2}$$

The 5kg mass of course moves with acceleration of only 4.9ms^{-2} .

Example 2: As the second example let me take a hollow cylinder that is rotating about its axis with a constant angular speed ω . Because of this rotation a mass *m* on the wall of the cylinder does not slip down (see figure 3). If the coefficient of friction between the cylinder wall and the mass is μ, what is the minimum value of w for this to happen?



Mass m inside a rotating cylinder. Its free body diagram when it is tuck to the wall of the cylinder (right).

Figure 3

For the mass not to slip, the maximum possible friction on it should be greater than the actual frictional force that holds it against its weight. Since the problem involves rotation we will use cylindrical coordinates. The free body diagram of the mass is as given in figure 3. The mass m experiences three forces when it is stuck to the wall of the cylinder. These forces are its weight *mg*, the normal reaction *N* of the cylinder and the frictional force *f* . In cylindrical coordinates the acceleration of the mass is

$$\left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z}$$

so that

$$m\left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi} + m\ddot{z}\hat{z} = -N\hat{r} + \left(f - mg\right)\hat{z}$$

Now

$$\left.\begin{array}{lll} r = R = \text{constant} & \Rightarrow & \dot{r} = \ddot{r} = 0 \\ z = \text{constant} & \Rightarrow & \ddot{z} = 0 \\ \dot{\phi} = \omega \ddot{\phi} = 0 & & \end{array}\right\} \quad \Rightarrow \quad -mR\omega^2\hat{r} + -N\hat{r} + (f - mg)\hat{z} = 0$$

which gives

$$N = mR\omega^2 \quad \Rightarrow \quad f_{\max} = \mu mR\omega^2 \quad \text{and} \quad f = mg$$

From this minimum angular speed $\omega_{\min}$ is calculated as follows.

$$f_{\max} > f \quad \Rightarrow \quad \mu mR\omega^2 > mg \quad \text{or} \quad \omega > \sqrt{\frac{g}{\mu R}}$$

Thus $\omega_{\min} = \sqrt{\frac{g}{\mu R}}$.

So far we have discussed one kind of frictional force where two solid bodies are in contact. We now learn to deal with the drag force which is experienced when a body is moving through gas or a liquid. This force arises due to viscosity of the fluid. To the lowest order in the velocity $\vec{v}$ of the moving body, the drag force is approximated by

$$\vec{F}_{drag} = -k\vec{v}$$

that is, it is a force in the direction opposite to the velocity and its magnitude is proportional to the speed. So the equation of motion in presence of drag force will read

$$\begin{aligned} m\ddot{\vec{r}} &= \vec{F}_{applied} - k\vec{v} \\ &= \vec{F}_{applied} - k\dot{\vec{r}} \end{aligned}$$

If we write it in its component form we have

$$\begin{aligned} m\ddot{x} &= F_x - k\dot{x} \\ m\ddot{y} &= F_y - k\dot{y} \\ m\ddot{z} &= F_z - k\dot{y} \end{aligned}$$

These formulae are valid when the speed of the object is not very large; at large speeds the drag force becomes proportional to the square of the velocity. The simplest example of the effect of drag is the falling raindrops. Although falling from great heights, they do not hit us with very large speed because of the drag force on them.

As an object falls vertically through a liquid/gas, the drag force on it increases with its speed. At a certain speed - when the drag force equals the weight of the object - it stops accelerating further and therefore moves with a constant speed. This speed is known as the terminal speed or terminal velocity. Assuming drag force to be linearly dependent on velocity, let us estimate the terminal speed of an object when it falls through a liquid of viscosity η. Let the vertically downward direction be y, then

$$m\ddot{y} = mg - k\dot{y}$$

But the object will stop accelerating, i.e. $\ddot{y} = 0$, after attaining the terminal speed. Thus at the terminal speed $\dot{y}_{term}$

$$mg = k\dot{y}_{term}$$

which gives

$$\dot{y}_{term} = \left(\frac{mg}{k}\right)$$

That is the terminal speed of the object. To estimate the terminal speed we need to know what k is. For a spherical object of radius a moving with low speeds, stokes formula gives the drag force to be

$$F_{drag} = -6\pi\eta a v$$

If the object is made of a material of density ρ, the terminal speed comes out to be

$$\dot{y}_{term} = \frac{2}{9}\left(\frac{a^2\rho g}{\eta}\right)$$

Let us estimate what will be the terminal speed of a rain drop of 2mm radius. With the viscosity of air $\eta = 1.8\times10^{-5}\ Ns/m^2$, we get

$$\dot{y}_{term} = \frac{2}{9}\times\frac{4\times10^{-6}\times10^{3}\times9.8}{1.8\times10^{-5}} \approx 485 m/s\ (1750\, km/h)$$

This is too high compared to the observed speeds of about 20 *kmph* to 5 *kmph*. Obviously the dependence of drag force on raindrops has higher power dependence on their speeds. In this lecture we will however restrict ourselves to those cases where the drag force depends linearly on the speed i.e. $\vec{F}_{drag} = -k\vec{v}$. We now solve examples involving such drag force.

Example 3 : An object is thrown in a fluid with initial speed v_0. Find its speed and the distance traveled by it as a function of time.

Assuming the motion to be in x direction, the equation of motion is

$$m\ddot{x} = -k\dot{x}$$

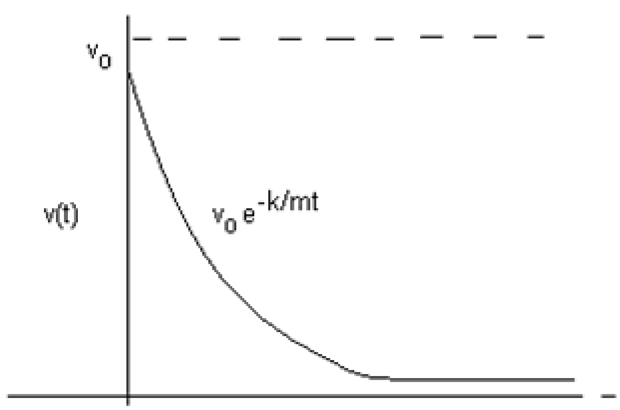
or

$$\frac{d}{dt}(\dot{x}) + \frac{k}{m}\dot{x} = 0$$

You can easily check that the solution is

$$\dot{x} = \mathrm{v}_o e^{-(k/m)t}$$

So that the speed initially is v_0 and it decreases exponentially with time. The plot of speed versus time looks like that given below

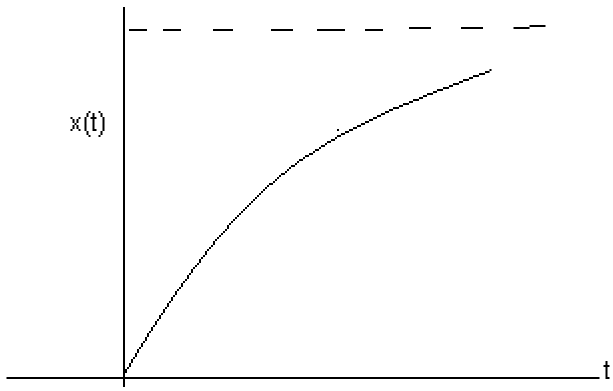


What about the distance traveled by the object? That is obtained by integrating the speed with respect to time and is

$$x(t) = \int_o^t \mathrm{v}_o e^{-(k/m)t'} dt'$$

$$= \frac{m\mathrm{v}_o}{k}\left[1 - e^{-(k/m)t}\right]$$

So that the distance traveled looks like



Thus as $t \to \infty$, the body will stop after traveling a distance of

$$\left(\frac{m v_o}{k}\right)$$

Of course as $k \to 0$, the distance becomes larger and larger.

Example 4 : We now consider one-dimensional motion of a particle which is moving under the influence of a constant applied force in a medium applying a drag force. Motion of a particle thrown up or falling down is one such example. The equation of motion in this case is

$$m\ddot{x} = F - k\dot{x}$$

$$\text{or } \frac{d}{dt}(\dot{x}) + \frac{k}{m}\dot{x} = \frac{F}{m}$$

Let us take the force to be F and the initial speed of the particle to be zero. Without the term F/m on the right-hand side, the solution of the equation above was $\dot{x} = v_o e^{-k/mt}$ which is, in the language of differential equations, the solution of the homogeneous equation i.e., equation with 0 on the right-hand side. To get the general solution, we add to the homogeneous solution the particular solution corresponding to $F \neq 0$. The particular solution is

$$\dot{x} = \frac{F}{k}$$

So that the general solution for the velocity is

$$\dot{x}(t) = \frac{F}{k} + v_o e^{-(k/m)t}$$

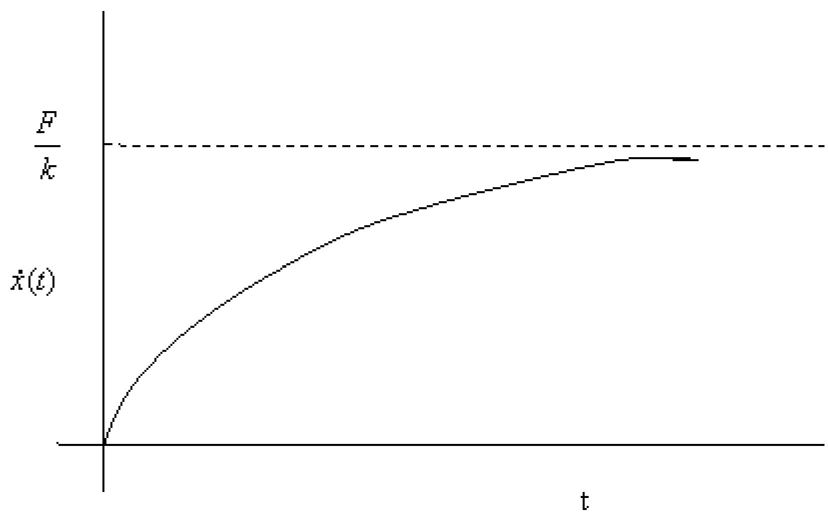
Here v_0 is some constant (not the initial velocity, which is given to be 0). If we start with $v(t=0)=0$ we get

$$v_o = -\frac{F}{k}$$

which gives

$$\dot{x}(t) = \frac{F}{k}[1 - e^{-(k/m)t}]$$

The plot of velocity versus time looks as follows



with the terminal speed being $\frac{F}{k}$. The next question we ask if the solution goes to the standard solution $\left(\frac{F}{m}t\right)$ of particle moving with a constant acceleration when k=0. From $\dot{x}(t) = \frac{F}{k}[1 - e^{-(k/m)t}]$ we get an answer of 0/0 so we have to be careful in taking $k = 0$. Recall that the solution was obtained by assuming $k \neq 0$ because we have been dividing by k. Thus for the $k=0$ case we should take the limit of $k \to 0$. Doing that we find

$$\dot{x}(t) = \frac{F}{k}(1 - 1 + \frac{k}{m}t - \frac{k^2}{m^2}t^2 \ldots\ldots$$
$$= \frac{F}{m}(t + O(k))$$

Now $k \to 0$ gives $\dot{x}(t) = \frac{F}{m}t$ which is the correct answer. We now calculate the distance *x(t)* traveled by the object as a function of time.

$$x(t) = \int_o^t x'(t')dt'$$
$$= \frac{F}{k}\int_o^t [1 - e^{-(k/m)t'}]dt'$$
$$= \frac{F}{k}\left[t - \frac{m}{k}\left(1 - e^{-(k/m)t}\right)\right]$$

You can see that t → ∞ the distance is given as

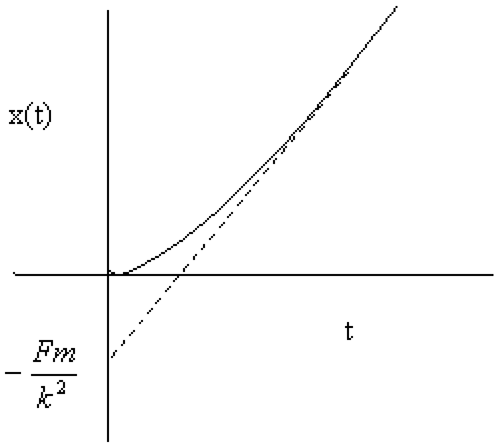
$$x(t) = \frac{F}{k}t - \frac{Fm}{k^2}$$

so at large times it increases linearly with the terminal speed.

For t → 0 it is

$$x(t) = \frac{F}{k}\left[t - \frac{m}{k}\left(1 - (1 - \frac{k}{m}t + \frac{1}{2}\frac{k^2}{m^2}t^2 + \ldots\ldots\right)\right]$$
$$= \frac{F}{k}\left[t - \frac{m}{k}\left(\frac{k}{m}t - \frac{1}{2}\frac{k^2}{m^2}t^2 + \ldots\ldots\right)\right]$$
$$= \frac{1}{2}\frac{F}{m}t^2$$

This is easily understood as initially there is no drag due to small initial speed and the distance is given by the formula for uniform acceleration. Combining the two limiting cases we see that the plot of x(t) versus time looks like



I'll leave it as an exercise that as $k \to 0$, we recover the familiar result $x(t) = \frac{1}{2}\frac{F}{m}t^2$ Also I would like you to solve for the velocity and height of a ball thrown up with an initial speed v 0 when drag of air is taken into account.

Next we analyze the effect of drag on the projectile motion in the gravitational field. In this case, we have a projectile shot with initial speed v_0 at an angle θ_0 from the horizontal and we want to find to subsequent motion. The equations of motion are (taking vertically up direction as the y-direction)

$$m\ddot{x} = -k\dot{x}$$
$$m\ddot{y} = -mg - k\dot{y}$$

We have already solved these equations above, so the speed and distance in the x-direction is given as

$$\dot{x}(t) = \mathrm{v}_o \cos\theta_o e^{-(k/m)t}$$

and $$x(t) = \frac{m\mathrm{v}_o \cos\theta_o}{k}[1 - e^{-(k/m)t}]$$

The equation of motion in the y-direction is

$$\frac{d}{dt}(\dot{y}) + \frac{k}{m}\dot{y} = -g$$

Its solution with the initial condition $\dot{y}(o) = \mathrm{v}_o \sin\theta_o$ is

$$\dot{y}(t) = -\frac{mg}{k}[1 - e^{-(k/m)t}] + \mathrm{v}_o \sin\theta_o e^{-(k/m)t}$$

I give you an exercise now: find at what time s $\dot{y} = 0$? Show that this time correctly goes to $\frac{\mathrm{v}_0 \sin\theta_0}{g}$ when $k = 0$. Integrating the speed, we get the height *y(t)* as a function of time. It is given as

$$y(t) = \frac{m\mathrm{v}_o \sin\theta_o}{R}[1 - e^{-(k/m)t}] - \frac{mg}{k}\left\{t - \frac{m}{k}[1 - e^{-(k/m)t}]\right\}$$

Now to get the trajectory one calculates *x(t)* and *y(t)* separately and plots *y* versus *x* . I give you some of these for a given v_o, θ_o but varying *k*. We take v_0 = 100m/s and $\theta_0 = 45^o$. For no drag situation we get the range *R = 1010m* and the highest point of the projectile to be at *h = 254m* . When a drag coefficient of *k = 0.1* is introduced we get *R = 495m* and *h = 175m* , a reduction of about *50%* in the range and *30%* in the height. For *k = 0.2* we get *R = 313m* and *h = 135m*, giving a further reduction of about *40%* in the range and *20%* in the height from the corresponding *k = 0.1* values. Notice when drag force is introduced, the range gets affected much more than the height. The corresponding trajectories are shown below.

One interesting question we may ask is: for zero drag the maximum range is obtained for θ = 45º. If we include drag, should the angle be larger than or less than 45º for obtaining maximum range? Since x-component of the velocity is now decreasing one intuitively feels that the projectile should be given larger speed in the x-direction for maximum range. Thus the projectile should be fired at an angle less than 45º. This is easily understood from the calculations presented above. As we saw in those calculations, for $k \neq 0$ the motion in y direction does not get affected as much as it does in the x-direction. This also suggests that for maximum range we fire the projectile at an angle slightly less than 45º giving it a lager velocity in x-direction. One can also think of it slightly differently. When the particle is shot up drag force is large (because of the initial speed) and also both the gravitational force and drag are working in the same direction. So the partial takes longer to move up the same height than it does in coming down. Since x-velocity is larger in the beginning, the projectile should cover as much distance as possible while ascending than when it is coming down (the x-component may well vanish by that time) This implies that θ_0 should be smaller than 45º.

What we have done so far is to include the simplest form of drag force in solving for the trajectories of motion. However, as the speed increases drag force may also include higher powers of velocity i.e. it may take the form

$$\vec{F}_{drag} = -k_1\vec{v} - k_2 v^2 \hat{v} + ...$$

where $\hat{v}$ is the unit vector in the direction of the velocity. This is written here to show that force is opposite to the velocity vector. In such cases the corresponding differential equation become non-linear in v and getting the solution becomes difficult, necessitating the use of numerical methods. Some problems though do allow analytic solutions. I end this lecture by giving you one such problem to solve.

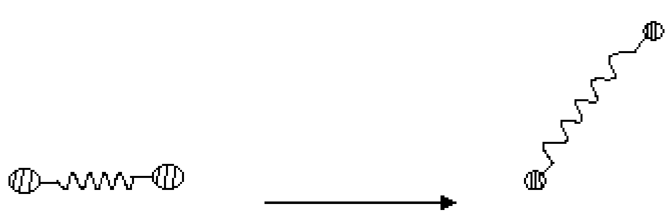
Exercise : Throw a ball up will initial velocity v_i and let the force of drag be $= -kv^2$. Find the final speed v_f of the ball when it hits the ground. Also find the height that it goes up to.

Lecture 14
Momentum

So far we have dealt with motion of single particles. Now we are going to make the situation slightly more difficult by letting two or more particles apply forces on one another either by coming in contact or from a distance, and see how we can describe their motion. In such a situation the motion become much more interesting. Let us take an example of only two particles interacting through a spring connected to them, as shown below.

During their motion any of the following could take place: the distance between them may change,

or their orientation may change,



or a combination of both these may occur. Now we wish to develop methods of dealing with such situations. We do this gradually by taking one step at a time. In this regard, we start by introducing the quantity *momentum* that plays a very important role in describing motion when more than one point particle are involved in the motion.

To understand the importance of momentum, let us do the following experiment. Take a cart moving on a frictionless horizontal plane and start putting mass into it; it may be dropped vertically in it (see figure 1 below).

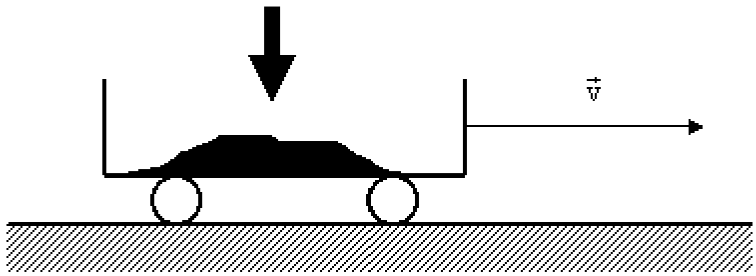


Figure 1

You will see that the cart starts slowing down. If we wish to keep it moving with the same velocity, we find that we have to apply a force on it

$$\vec{F} = \left(\frac{\Delta M}{\Delta t}\right)\vec{v}$$

Compare this with the standard form of Newton's IInd law where we put

$$\vec{F} = M\,\vec{a} = M\,\frac{\Delta \vec{v}}{\Delta t}$$

So we see that whether the mass is changed and the velocity kept constant, or the velocity is changed and the mass is kept constant, we have to apply a force to a body. Thus in general

$$\vec{F} = \left(\frac{\Delta M}{\Delta t}\right)\vec{v} + M\,\frac{\Delta \vec{v}}{\Delta t}$$

(We have ignored the second-order term $\frac{\Delta M \Delta \vec{v}}{\Delta t}$ right now assuming that both the mass and the velocity are varying continuously). Therefore

$$\vec{F} = \frac{\Delta(M\vec{v})}{\Delta t} = \left(\frac{dM\vec{v}}{dt}\right)$$
$$= \frac{d\,\vec{p}}{dt}$$

and this defines for us a quality called the momentum denoted above by $\vec{p}$. By definition

$$\vec{p} = M\vec{v}$$

The force applied on a body or a system of particles is then the rate of change of their total momentum, i.e.

$$\vec{F} = \left(\frac{d\vec{p}}{dt}\right)$$

where $\vec{p}$ now refers to the momentum of the system made up of a collection of particles. In the example taken above, we have to apply a force to keep the cart moving with a constant velocity because as the mass falls in the cart and starts moving with same velocity as the cart, the total momentum of the system - the cart and the mass in it - increases. In writing the definition of the momentum above, we have implicitly assumed that all the particles of the system, with total mass M, are moving with the same velocity. However, if the system is made up of *N* particles, each one being of different mass m_i ($i = 1$ *to* N) and also moving with a different velocity $\vec{v}_i$, the total momentum of the system will be given as

$$\vec{p} = \sum_i m_i \vec{v}_i$$

A fundamental property of momentum is now follows from the definition of force in terms of momentum. If the total force acting on a system of particles is zero, the total momentum of the system does not change with time. To see it clearly let us go back to the two particles connected by a spring (see figure 2 below). There we have

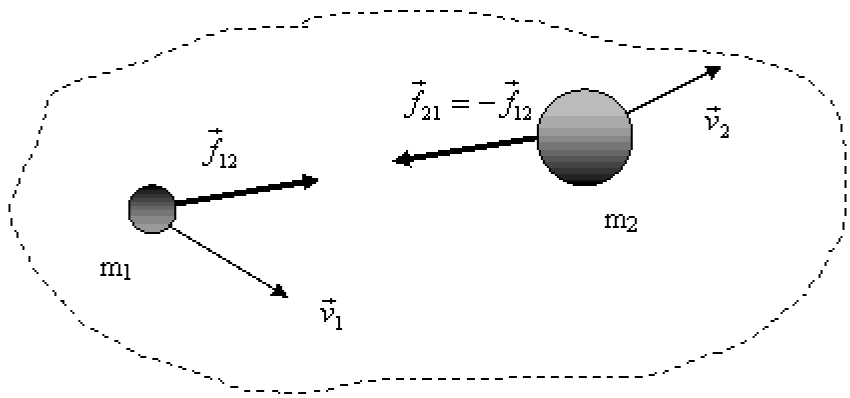
$$m_1 \frac{d\vec{v}_1}{dt} = \vec{f}_{12}$$

for particle 1 and

$$m_2 \frac{d\vec{v}_2}{dt} = \vec{f}_{21}$$

for particle 2. Here $\vec{f}_{12}$ is the force on particle 1 applied by particle 2. Similarly $\vec{f}_{21}$ is the force on particle 2 applied by particle 1. By Newton 's third law

$$\vec{f}_{21} = -\vec{f}_{12}$$



Two particles of different masses moving with different velocities. They are applying equal and opposite force on one another.

Figure 2

This immediately results in

$$m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} = \frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) = 0$$

So no matter how these particles move - their individual velocities $\vec{v}_1$ or $\vec{v}_2$ may change - but as long as there is no other force on the system and Newton's third law is obeyed we are going to have

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \text{constant}$$

The equation above expresses the principle of momentum conservation - which is a fundamental principle of physics - in its simplest form.

Let us understand this result. If we consider both the particles together as one system, indicated by the dashed line enclosing them in the figure above, there is no force on this system. This is because although each particle is acted upon by a force applied by the other particle, on the system as a whole these two forces act in opposite directions and cancel each other, resulting in a zero net force on the system. As such the momentum of the system does not change. Thus we conclude: ***If the net force acting upon a system of two particles vanishes, their total momentum does not change with time*** . Let us now see what happens when we apply forces on each particle also. In that case we have

$$m_1 \frac{d\vec{v}_1}{dt} = \vec{F}_{ext1} + \vec{f}_{12}$$

$$m_2 \frac{d\vec{v}_2}{dt} = \vec{F}_{ext2} + \vec{f}_{21}$$

$$= \vec{F}_{ext1} - \vec{f}_{12}$$

which gives

$$m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} = \vec{F}_{ext1} + \vec{F}_{ext2}$$

$$\text{or} \qquad \frac{d\vec{p}}{dt} = \vec{F}_{total}$$

Again we see that no matter how the individual velocities change, the total momentum changes according to the equation

$$\left(\frac{d\vec{p}}{dt}\right) = \vec{F}_{total}$$

Let us now generalize this result to a system of many particles (say N). Then we have for the i^{th} particle

$$m_i \frac{d\vec{v}_i}{dt} = \vec{F}_{exti} + \sum_{\substack{j \\ j \neq i}} \vec{f}ij$$

Where $\vec{F}_{ext\,i}$ is the external force on the i^{th} particle and $\vec{f}_{ij}$ is the force applied on i^{th} particle due to j^{th} particle. Summing it over i gives

$$\sum_i m_i \frac{d\vec{v}_i}{dt} = \sum_i \vec{F}_{ext\,i} + \sum_{\substack{i\,j \\ i \neq j}} \vec{f}\,i\,j$$

Now we can write

$$\sum_{\substack{ij \\ i \neq j}} \vec{f}\,i\,j = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} (\vec{f}\,ij + \vec{f}\,\,j\,i)$$

But by Newton 's third law $\vec{f}_{ij} = -\vec{f}_{ji}$ which when substituted in the equation above gives

$$\sum_{\substack{ij\\ i\neq j}} \vec{f}ij = 0$$

and $$\sum_i m_i \frac{d\vec{v}_i}{dt} = \sum_i \vec{F}_{ext\,i}$$

i.e., the total momentum of a system of particles changes due to only the net outside force applied on the system; the interaction between particles does not affect their total momentum. And if $\vec{F}_{ext=0}$ i.e., there is no external force on the system,

$$\left(\frac{d\vec{p}}{dt}\right) = 0$$

which means that the total momentum of the system is a constant. That is the statement of conservation of momentum. We will see later that when combined with the principle of conservation of energy, it becomes a powerful tool for solving problem in mechanics. For the time being let us use this principle to develop some intuitive feeling about motion of a collection of particles; looking at it as a single mass.

We now introduce you to the concept of the centre of mass (CM). To do this, let us look at the equation of motion

$$\left(\frac{d\vec{p}}{dt}\right) = \vec{F}_{ext}$$

which is equivalent to

$$\frac{d}{dt}\sum m_i \frac{d\vec{v}_i}{dt} = \vec{F}_{ext\,i}$$

Since total mass of a collection of particles remains the same, we can divide and multiply the left-hand side of the equation above by the total mass to rewrite it as

$$M\frac{d}{dt}\sum \frac{m_i \dfrac{d\vec{v}_i}{dt}}{M} = \vec{F}_{ext\,i}$$

Since $\vec{v}_i = \dfrac{d\vec{r}_i}{dt}$, where $\vec{r}_i$ is the position of the i^{th} particle, the above equation can also be written as

$$M\frac{d^2}{dt^2}\sum_i m_i\left(\frac{m_i\vec{r}_i}{M}\right) = \vec{F}_{ext\,(total)}$$

Now we introduce the position vector $\vec{R}_{CM}$ for the centre of mass by writing

$$\vec{R}_{CM} = \frac{m_i \vec{r}_i}{M}$$

so that the equation of motion looks as follows

$$M \frac{d^2 \vec{R}_{CM}}{dt^2} = \vec{F}_{ext(total)}$$

Now we interpret this equation: It says that irrespective of the interaction between the particles and their relative motion, the centre of mass of a collection of particles would always move as if it were a point particle of total mass *M* moving under the influence of the sum of externally applied forces on each particle, i.e., the total external force. I caution you that the equation above does not imply that all the particles are moving the same way. All it says is that they move in such a way that the motion of their CM is described as if the *CM* was a particle of mass M.

Let us take an example.

Example 1: Suppose a bomb dropping vertically down explodes in mid air and breaks into three parts. Let the mass of the bomb be *m* and those of three pieces $\frac{m}{6}, \frac{m}{3}$ and $\frac{m}{2}$, respectively. If the heaviest piece falls *10m* to the east and the lightest piece *12m* south of where the unexploded bomb would have dropped, where does the third piece fall?

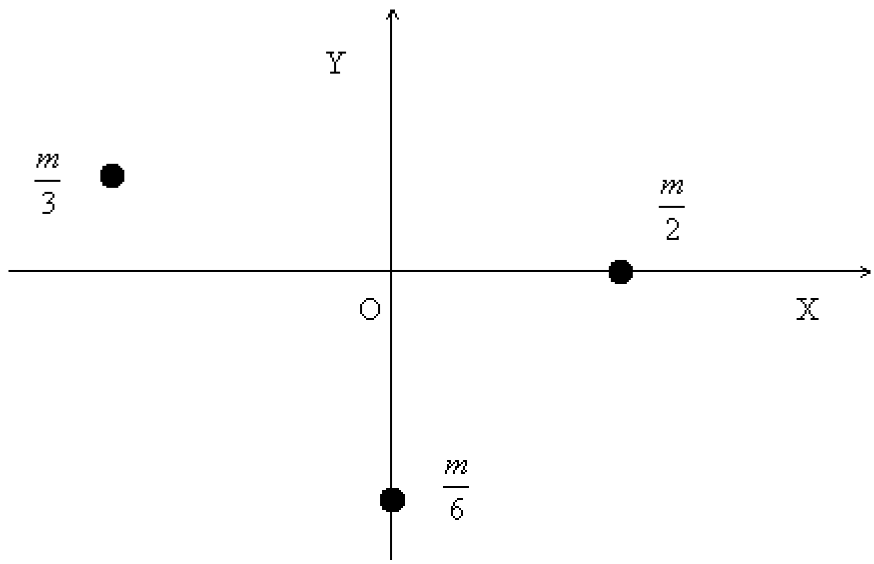
Since $M\ddot{\vec{R}}_{CM} = \vec{F}_{ext(total)}$, the CM keeps on moving - even after the bomb breaks - vertically down as if it were a point mass of mass *M* falling under gravity. Thus the CM hits the ground where the unexploded bomb would have fallen. Let us take this point to be the origin with east side being the positive x-axis and the north side the positive y-axis. Then $\vec{R}_{CM} = 0, \vec{r}_{m/2} = 10\hat{i}, \vec{r}_{m/6} = -12\hat{j}$ after the bomb pieces having moved for equal times. By definition of the centre of mass we have

$$m\vec{R}_{CM} = \frac{m}{6} \times -12\hat{j} + \frac{m}{3}\vec{r}_{m/3} + \frac{m}{2} \times 10\hat{i}$$

With $\vec{R}_{CM} = 0$, this gives

$$\vec{r}_{m/3} = -15\hat{i} + 6\hat{j}$$

Relative positions of the three pieces are shown in figure 3 below, with the centre of mass at the origin.



Positions of the three pieces of the bomb on ground

Figure 3

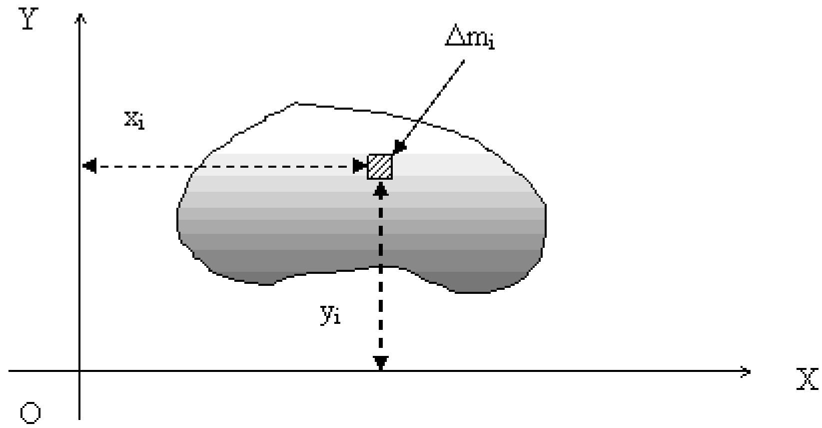
You see that having the knowledge about the position of the other two pieces, we have got the position of the third piece without the knowing anything about the forces generated during the explosion and therefore without solving any equation of motion. That is the power of the momentum conservation principle. I will leave it for you to think which component of momentum is conserved in this case. Would that component be conserved if drag force were included?

Other familiar examples of momentum conservation are a gun recoiling when fired, two persons on roller seats pushing each other and consequently moving away from each other. Look around and you will find many such examples of momentum conservation.

I now discuss a little about calculation of the centre of mass of a mass distribution. Calculation of the centre of mass is similar to calculating the centroid of an area (lecture 7), except that the area is now replaced by mass. For finite masses at given positions, the definition of centre of mass given above is used directly. For a mass distribution in three-dimensions, we calculate all three components of the poison of the centre of mass. These are given as

$$X_{CM} = \frac{\sum_i x_i \Delta m_i}{m} = \frac{\int x dm}{m}, \quad Y_{CM} = \frac{\sum_i y_i \Delta m_i}{m} = \frac{\int y dm}{m} \quad \text{and} \quad Z_{CM} = \frac{\sum_i z_i \Delta m_i}{m} = \frac{\int z dm}{m}$$

where *dm* is a small mass element at the position (*x*,*y*,*z*) in the mass distribution (see figure 4 below).



An element of mass Δm_i and its x- and y-coordinates

Figure 4

We are now going to change the topic a bit and ask how we describe a system where a large force acts for very short durations. A cricket bat striking a ball, a hammer hitting a nail, a person jumping on a floor and coming to sudden stop and a carom striker hitting a coin, or collisions in general, are examples of such forces in operation. In these cases it is not meaningful to talk about the force as a function of time because the time span over which the force acts is very-very short. Further, the force varies a great deal over this short time-interval, as I show in an example below. It is therefore better to describe the overall impact of the force in terms of the momentum change it causes to the system. This is given by the integral of the force over the time that it operates. Thus $\Delta\vec{p} = \int \vec{F} dt$ describes the effect of the force on the system. The integral $\int \vec{F} dt$ is known as the impulse and denoted by the symbol *J*. Obviously the momentum change of a system equals the impulse given to it. We now discuss these ideas with the help of an example, that of a ball hitting a wall or any other hard surface.

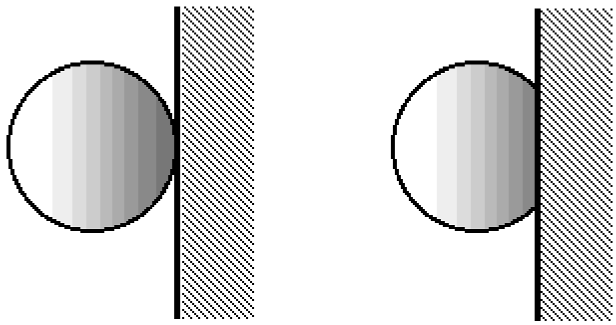
Let us ask what happens when a ball hits a wall or we jump on the floor. If the ball hitting the wall reflects back, that means that the wall has applied a force on the ball so that

$$\Delta\vec{p} = \int \vec{F}\, dt$$

If the time of contact between the ball and the wall is Δt seconds then the average force is

$$\vec{F}_{average} = \frac{\Delta\vec{p}}{\Delta t}$$
$$= \frac{J}{\Delta t}$$

But the real force varies greatly from the average force. We show that now. Take the model of the ball as following Hooke's law so that if it is compressed by *x* by the wall, it applies a force *kx* on the wall and consequently experiences an equal force in the opposite direction (see figure 5 below).



A ball hitting a wall and getting compressed by amount x

Figure 5

Since the force on the ball follows Hooke's law, the ball performs a simple harmonic motion, its compression is given by $x = A\sin\,\omega t$, where A is the maximum compression and $\omega = \sqrt{\frac{k}{m}}$.

From time $t = 0$, when the ball comes in and touches the wall, it takes $t = \frac{\pi}{\omega}$ time (half a cycle) before leaving the wall. The force during this time is given as

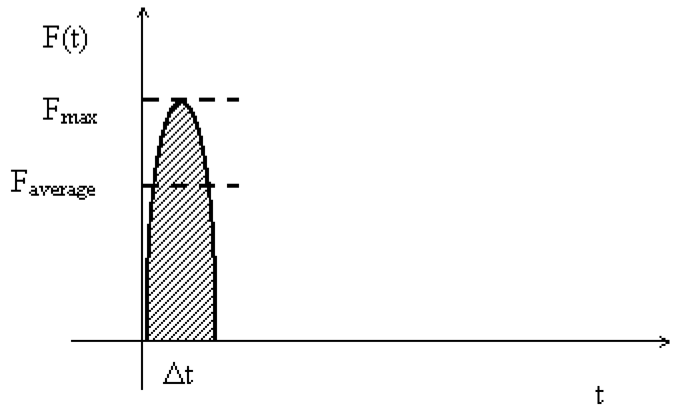
$$\begin{aligned} F &= kx \\ &= kA\sin\,\omega t \end{aligned}$$

Since for a hard ball *k* is very large, $\omega = \sqrt{\frac{k}{m}} >> 1$. So by the time the ball comes back, the force varies with time as shown in the figure 6 below. Here the maximum force F_{max} is given by *kA* and $\Delta t = \frac{\pi}{\omega}$. In the figure we show both F_{max} and $F_{average}$. The latter is calculated as

$$F_{average} = \frac{1}{(\pi/\omega)} \int_0^{\pi/\omega} kA\sin\,\omega t\,dt = \frac{2kA}{\pi}$$

or

$$F_{average} = \frac{2F_{max}}{\pi}$$



Variation of the force on a ball with time as it hits a hard surface at t=0, and gets reflected from it Δt time later. The area under the curve gives the impulse. Both F_{max} and $F_{average}$ are also shown.

Figure 6

So you see that over this short period force varies a great deal and is hardly ever near the average force that we calculated. The discussion above has been in terms of a model of the force; the exact force will be different this model and so the variation could be even larger than that shown. It is in such situations, when a strong force is applied over a very short time period, that it is much more meaningful to talk of the total momentum change of a particle than the force $\vec{F} = \left(\frac{d\vec{p}}{dt}\right)$. Further, in such cases, we generally observe only the initial & final momentum and are hardly concerned about the finer details. It is this change

$$\Delta\vec{p} = \int \vec{F} dt$$

In the momentum that is known as the impulse. So in the ball rebounding from a hard surface with the same speed as it comes in with, the impulse is $-2\vec{p}_i$, where $\vec{p}_i$ is the initial momentum of the ball. So instead of talking of the force applied by the ball on the surface, we say that the ball has imparted momentum to the surface it hit. The amount of momentum transferred is equal to the impulse. This has interesting application in calculating the force on a surface when there are many-many particles continuously hitting a surface, for example molecules in a vessel hitting its walls from inside.

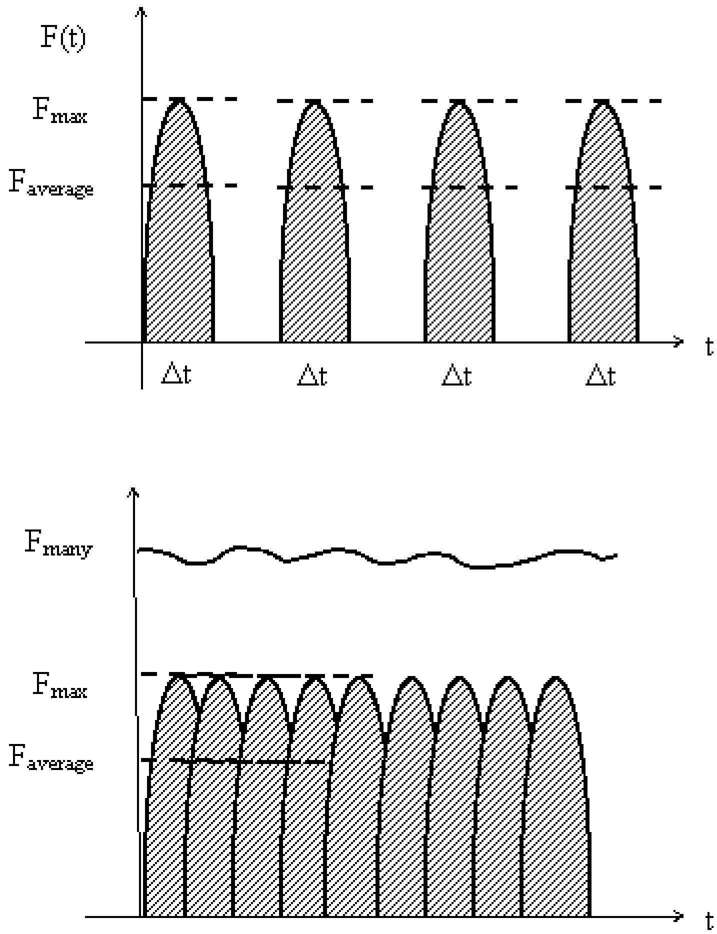
We show two situations in figure 7 below. The upper figure shows the variation of force on a wall when particles hit a surface at some time interval. The lower one, on the other hand, shows the situation when particles hit continuously. In the first case the force on the surface due to the particles hitting it varies pretty much like the force due to each particle itself. In the second case, however, the force at any instant is given as the sum of the forces applied by each particle at that time. This gives an almost constant force F_{many} as shown in the figure. The value of this force is calculated as follows. Let each particle hitting the surface impart an impulse *J* to it. If on an average there are *n* particles per second hitting the surface, then in time Δt the momentum transferred to the surface will be $(n\Delta t)J$. The force F_{many} will then be given as

$$F_{many} = \frac{(n\Delta t)J}{\Delta t}$$

Since $F_{average} = \frac{J}{\Delta t}$, the force above can also be written as

$$F_{many} = (n\Delta t)F_{average}$$

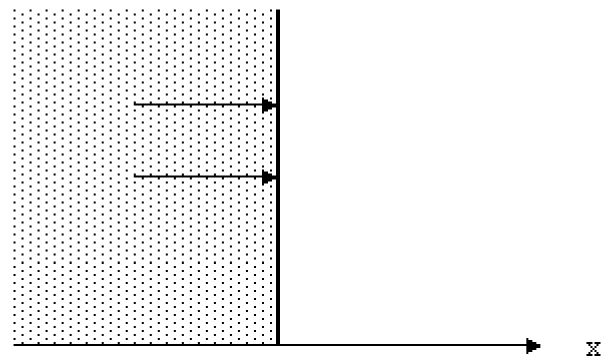
Thus when a stream of particles hits a surface, the force applied by them to the surface equals ***the number of particles striking in time Δt times the average force applied by each one*** of them, a result that you could have anticipated. This is precisely what happens when a jet of water or flowing mass hits another object.



Force on a surface when a stream of particles is hitting it. Upper figure shows the force when particles come one at a time whereas the lower one shows the force when the partcles are hitting the surface almost continuously.

Figure 7

As an example let us calculate the pressure of a gas filled in a container. Let the mass of each molecule be m and let their average speed be v . The number density of the molecules in the gas is taken to be n . Now consider a surface of the container perpendicular to the x-axis. (see figure 8).



Molecules of a gas hitting the wall perpendicular to the x-axis.

Figure 8

Each molecule, when reflected from the wall imparts a momentum equal to $2mv_x$ to the wall. The average number of molecules hitting are *A* of the wall per unit time will be half of those contained in a cylinder of base area *A* and height v_x (the other half will be moving in the other direction). This comes out to be $\frac{Av_x n}{2}$. Thus from the formula derived above the force on the wall applied by these molecules is

$$F = Anmv_x^2$$

which gives the pressure

$$p = \frac{F}{A} = nmv_x^2$$
$$= \frac{1}{3}nmv^2$$

This is a result you are already familiar with kinetic theory of gases. But now you know how it comes out. Having done this problem we now deal with another very interesting application of the momentum-force relationship, known as the variable mass problem.

So far we have been dealing with particles of fixed masses. Let us now apply the equation $\left(\frac{d\vec{p}}{dt}\right) = \vec{F}$ to a problem when the mass of the system under consideration varies with time. The most famous example of this is the rocket propulsion.

Let a rocket with mass *M* at time *t* be moving with velocity $\vec{v}$. A small mass Δm with velocity $\vec{u}$ comes and gets stuck with it so that the rocket now has mass $M + \Delta m$ and moves with a velocity $\vec{v} + \Delta\vec{v}$ (see figure 9 below) after a time interval of Δt. We want to find at what rate does the velocity of the rocket increase? *We point out that the word rocket has been used here to represent any system with variable mass .*

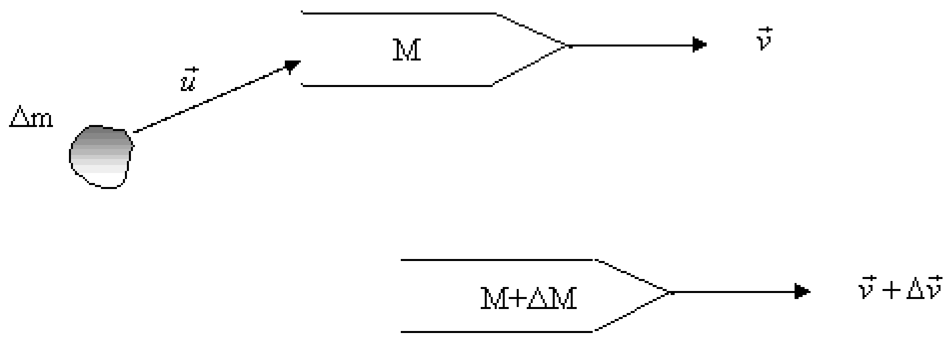


Figure 9

Let us write the momentum change in time interval Δt and equate this to the total external force on the system (that is the sum of external forces acting on M and Δm) times Δt. That gives

$$(M + \Delta m)(\vec{v} + \Delta\vec{v}) - M\vec{v} - \Delta m\vec{u} = \vec{F}_{ext}\Delta t$$

or

$$M\Delta\vec{v} = \Delta m\left(\vec{u} - \vec{v}\right) + \vec{F}_{ext}\Delta t - \Delta m\Delta\vec{v}$$

$(\vec{u} - \vec{v})$ is nothing but the relative velocity $\vec{u}_{rel}$ of the mass Δm with respect to the rocket.

Dividing both sides of the equation above by Δt then leads to

$$\frac{M\Delta\vec{v}}{\Delta t} = \frac{\Delta m}{\Delta t}\vec{u}_{rel} + \vec{F}_{ext} - \frac{\Delta m\Delta\vec{v}}{\Delta t}$$

We now let $\Delta t \to 0$. In this limit $\frac{\Delta m\Delta\vec{v}}{\Delta t}$ also goes to zero for continuously varying mass. Further, $\frac{dm}{dt} = \frac{dM}{dt}$, the rate of change of the mass of the rocket. Thus the equation for the velocity of a rocket is

$$M\frac{d\vec{v}}{dt} = \frac{dM}{dt}\vec{u}_{rel} + \vec{F}_{ext}$$

Note that both the mass and velocity are now functions of time. For a rocket $\frac{dM}{dt} < 0$ and $\vec{u}_{rel} < 0$ so that $M\frac{d\vec{v}}{dt} > 0$. It is this term that provides the thrust to the rocket. As pointed out above, ***although this equation has been derived keeping rocket in mind, it is true for any system with variable mass*** .

Example: We now solve a simple problem involving the rocket equation. A rocket is fired vertically up in a gravitational field. What is its final velocity assuming that the rate of exhaust and its relative velocity remain unchanged during the lift off?

The motion of rocket is one-dimensional. We take the vertically up direction to be positive. Then we have $\vec{u}_{rel} = -u$ where *u* is a positive number. Therefore the rocket equation takes the form

$$M\frac{dv}{dt} = -\frac{dM}{dt}u - Mg$$

which gives

$$dv = -\frac{dM}{M}u - g\,dt$$

or

$$v_f = u\ln\left(\frac{M_i}{M_f}\right) - gt_f$$

Here we have taken the initial time and initial velocity both to be zero. Even after the fuel has all been burnt, we see if we observe the rocket time *t* after being fired, its velocity will be given by the formula

$$v(t) = u\ln\left(\frac{M_i}{M_f}\right) - gt$$

assuming *g* to be a constant.

Finally, although the momentum-force equation can provide answers for the velocities, I would like to urge you to always think about how the internal forces that generate momenta in opposite directions are generated. That helps in understanding the underlying physics better.

For example in the rocket problems, we say that $\frac{dM}{dt}\vec{u}_{rel}$ provides the thrust to make the rocket move forward. But think about what generates this force? The answer is as follows. In a closed container, gas pressure applies force in all directions and these forces cancel each other. But when a hole is made from where the gas can escape, the force in the opposite direction is unbalanced; and that is what makes the rocket move. If you understand this, you should e able to answer the following question. If we take a closed box with vacuum inside and punch a hole in it. Which way will it move?

We conclude this lecture by summarizing what we have learnt. We studied the conservation of momentum and a related concept of the centre of mass. Using momentum, we then calculated the force on a surface being hit by a stream of particles, or jet of water. Finally we learnt about the variable mass problem and applied it to a rocket taking off. In the coming lecture we will use the conservation of momentum principle along with the conservation of energy and see how this combination becomes a powerful tool in solving mechanics problems.

Lecture 15 & 16
Work and Energy

You have been studying in your school that we do work when we apply force on a body and move it. Thus performing work involves both the application of a force as well as displacement of the body. We will now see how this definition comes about naturally when we eliminate time from the equation of motion.

The question that immediately comes to mind is why should we eliminate time from the equation of motion. This is because when we follow the motion of a particle, we are usually interested in velocity as a function of position. Secondly, if we write the equation of motion in terms of time derivatives, it may make the equation difficult to solve. In such cases eliminating time from the equation of motion helps in solving the equation. Let us see this through an example.

Example: Consider the motion of a particle in a gravitational field of mass M . Gravitational force on a mass m is in the radial direction and is given as

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\hat{r}$$

Since the force in the radial direction, it is better to write the equation of motion in spherical polar coordinates. For simplicity we consider the motion only along the radial direction so that the equation of motion is written as

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2}$$

As you can see, integrating this equation to get r(t) as a function of time is very difficult.

On the other hand, let us eliminate time from the equation by using chain rule of differentiation to get

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{dr}{dt}\frac{d}{dr}\left(\frac{dr}{dt}\right) = v\frac{dv}{dr},$$

where $v = \dot{r}$ is the velocity in the radial direction. This changes the equation of motion to

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$

This equation is very easy to integrate and gives $v = \dot{r}$ as a function of *r*, which can hopefully be further integrated to get *r* as a function of time. Now we go back to what I had said earlier that the definition of work and energy arises naturally when we eliminate time from the equation of motion. Let us do that first for one dimensional case and analyze the problem in detail.

Work and energy in one dimension

The equation of motion in one-dimension (taking the variable to be *x*, and the force to be *F*) is

Let us again eliminate time from the left-hand using the technique used above

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right) = v\frac{dv}{dx}$$

to get

$$mv\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2\right) = F(x)$$

On integration this equation gives

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int_{x_i}^{x_f} F(x)dx$$

where x_i and x_f refer to the initial and final positions, and v_i and v_f to the initial and final velocities, respectively. We now interpret this result. We define the kinetic energy of a particle of mass m and velocity v to be

$$Kinetic\ energy = \frac{1}{2}mv^2$$

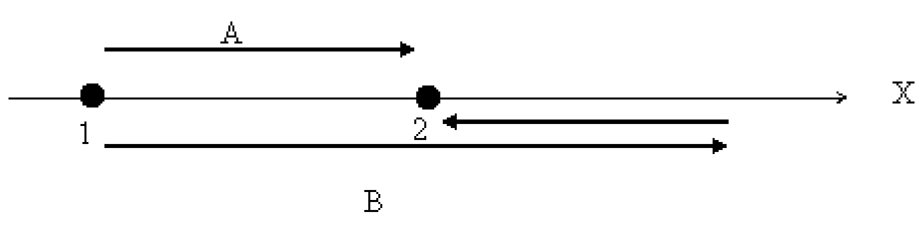
and the work done in moving from one position to the other as the integral given above

$$Work\ done = \int F(x)dx$$

With these definitions the equation derived above tells us that work done on a particle changes its kinetic energy by an equal amount; this known as the *work-energy theorem* .

You may ask: how do we know this equation to be true and consistent with our observations? This is the question that was asked in the early eighteenth century when it was not clear how to define energy, whether as mv or as mv^2 ? The problem with the definition as mv is that if two particles moving in the opposite directions have their energies canceling each other and if they collide, they stop and all the energy is *lost* **.** On the other hand, defining it proportional to v^2 makes their energies add up and noting is lost during collision; the energy just changes form but is conserved. Experimental evidence for the latter was found by dropping weights into soft clay floors. It was found that by increasing the speed of the weights by a factor of two made them sink in a distance roughly four times more; increase in the speed by a factor of three made it nine times more. That was the evidence in favor of kinetic energy being proportional to v^2 .

Potential energy: Let us now define another related energy known as the *potential energy* . This defined for a force field that may exist in the space, for example the gravitational field or the electric field. Before doing that we first note that even in one dimension, there are many different ways in which one can go from point *1* to point *2* . Two such paths are shown in the figure below.



On path A the particle goes directly from point *1* to *2* , whereas on path B it goes beyond point *2* and then comes back. The question we now ask is if the work done is always the same in going from point *1* to point *2*. This is not always true. For example if there is friction, the work done against friction while moving on path B will be more that on path A. If for a force the work done depends on the path, potential energy cannot be defined for such forces. On the other hand, if the work W_{12} done by a force in going from *1* to *2* is independent of the path, it can be expressed as the difference of a quantity that depends only on the positions x_1 and x_2 of points *1* and *2* (***Question:*** *If the work done is independent of path, what will be the work done by the force field when a particle comes back to its initial position?*). We write this as

$$W_{12} = \int_{x1}^{x2} F(x)\,dx = -U(x_2) + U(x_1)$$

and call the quantity *U(x)* the potential energy of the particle. We now interpret this quantity. Assume that a particle is in a force field *F(x)* . We now apply a force on the particle to keep it in equilibrium and move it very-very slowly from point *1* to *2*. Obviously the force applied by *us* is - *F(x)* and the work done by us in taking the particle from *1* to *2,* while maintaining its equilibrium, is

$$-\int_{x1}^{x2} F(x)\,dx = -W_{12} = U(x_2) - U(x_1)$$

Thus for a given force field, the potential energy difference $U(x_2) - U(x_1)$ between two points is the *work done by us* in moving a particle, keeping it in equilibrium, from *1* to *2* . Note that it is the work done by us - and not by the force field - that gives the difference in the potential energy. By definition, the work done by the force field is negative of the difference in the potential energy. Further, it is the difference in the potential energy that is a physically meaningful quantity. Thus is we want to define the potential energy *U(x)* as a function of *x* , we must choose a reference point where we take the potential energy to be zero. For example in defining the gravitational potential energy near the earth's surface, we take the ground level to be the reference point and define the potential energy of a mass *m* at height *h* as *mgh* . We could equally well take a point at height h_0 to be the reference point; in that case the potential energy for the same mass at height *h* would be $mg(h - h_0)$. Let us now solve another example.

Example: A particle is restricted to move along the x-axis and is acted upon by a force $F(x) = \dfrac{1}{x^2 + a^2}$. Find its potential energy.

We first note that the force is always acting towards the positive x-direction. Thus when we move the particle, we will have to do positive work when taking it towards the negative direction. Thus we expect the potential energy to increase as *x* becomes more and more negative. By definition

$$U(x_2) - U(x_1) = -\int_1^2 F(x)\,dx$$

$$= -\int_{x1}^{x2} \frac{1}{x^2 + a^2}\,dx$$

$$= \tan^{-1}\left(\frac{x_1}{a}\right) - \tan^{-1}\left(\frac{x_2}{a}\right)$$

Now we choose our reference point. If we choose $U(x_1 = \infty) = 0$, the potential energy is given as

$$U_1(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{a}\right)$$

On the other hand if we choose $U(x_1 = 0) = 0$, we get

$$U_2(x) = -\tan^{-1}\left(\frac{x}{a}\right)$$

The two energies are shifted with respect to one another by a constant so that the difference in the potential energy between two points is the same for both the forms, as pointed out earlier. The potential energy is lowest for $x = \infty$ and increases as we move towards left and becomes largest for $x = -\infty$. This is precisely what we had anticipated above on the basis of the meaning of potential energy

Conservation of energy: Having defined potential energy we now combine it with the work energy theorem to come up with another very important conservation principle: that of *conservation of energy* . This is obtained as follows. By the work energy theorem

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = W_{12} = \int_{x1}^{x2} F(x)\,dx$$

and by definition of the potential energy

$$W_{12} = -U(x_2) + U(x_1)$$

Combining the two equations we get

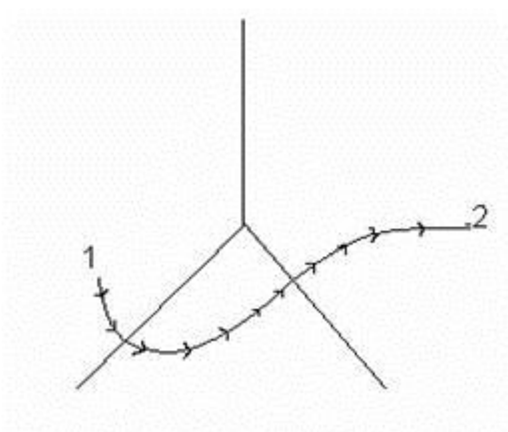
$$\frac{1}{2}mv_2^2 + U(x_2) = \frac{1}{2}mv_1^2 + U(x_1)$$

This equation means that if a particle moves in a force field where the work done by the force does not depend on the path taken, the sum of its kinetic and potential energy remains unchanged from one point to another. The sum of the kinetic and potential energy is known as the total mechanical energy. Thus in a force field for which the potential can be defined, total mechanical energy is conserved. Such force fields, where the total mechanical energy is conserved, are therefore known as conservative force fields. Thus whereas the example above is a conservative force field, frictional force is not. ***Question:*** *If the potential energy is explicitly time-dependent, is the total energy conserved?*

We now move on to generalize and discuss these concepts in three-dimensions.

Work and energy in three dimensions

As we already know, work is defined as the scalar product of the force and displacement vector. Thus if a particle moves under the influence of a force field $\vec{F}(\vec{r})$ from point 1 to point 2 along the path shown below, the total work is calculated as the sum of partial work done when the particle moves a vanishingly small distance $\Delta\vec{l}$ along the arrows shown below in the figure.



Thus the total work done in gives as

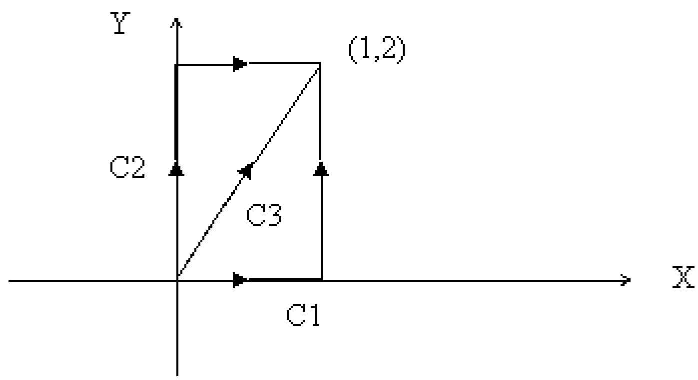
$$W_{12} = \int_{C(12)} \vec{F}(\vec{r}) \cdot d\vec{l}$$

where *C(12)* indicates that the particle is moving along the curve *C* from point *1* to *2* . Writing the dot product explicitly, we get

$$W_{12} = \int_{C(12)} F_x(x,y,z)dx + \int_{C(12)} F_y(x,y,z)dy + z + \int_{C(12)} F_z(x,y,z)dz$$

where F_i $(i = x,y,z)$ indicates the ith component of the force and *x*, *y* and *z* are varied along the curve. Let us do an example of calculating the work in this manner in two-dimensions.

Example: Consider two force fields (a) $\vec{F}(x,y) = x\hat{i} + y\hat{j}$, and (b) $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$ in the x-y plane. Calculate the work done by these forces when a particle moves from the origin to (1,2) along the three paths C1, C2 and C3 shown in the figure below. On C1 the particle goes along the x-axis first and then moves parallel to the y-axis; on C2 it travels along the y-axis first and then parallel to the x-axis and on C3 it moves along the diagonal.



The work done is given by the formula

$$W_{12} = \int_{C(12)} F_x(x,y,z)dx + \int_{C(12)} F_y(x,y,z)dy$$

Along C1 $y = 0$, $dy = 0$ while moving along the x-axis whereas $x = 1$ and $dx = 0$ when the particle travels parallel to the y-axis. Thus the work done along C1 is

$$W_{12}(C1) = \int_0^1 F_x(x, y=0)dx + \int_0^2 F_y(x=1, y)dy$$

Similarly work done along C2 is given as

$$W_{12}(C2) = \int_0^2 F_y(x=0, y)dy + \int_0^1 F_x(x, y=2)dx$$

For path C3, we have $y = 2x$ so that $dy = 2dx$. Therefore we substitute $y = 2x$ in the functions giving the force and replace dy by $2dx$. As a result, the final integration is over x only with x varying from 0 to 1 . Thus the work done is

$$W_{12}(C3) = \int_{C3} F_x(x,y)dx + \int_{C3} F_y(x,y)dy$$
$$= \int_0^1 F_x(x, y = 2x)dx + 2\int_0^1 F_y(x, y = 2x)dx$$

We are now ready to work out the work done by force in (a) and (b) (I would like you to plot these force fields and leave it as an exercise for you). For the force in (a) we get

$$W_{12}(C1) = \int_0^1 xdx + \int_0^2 ydy = \frac{5}{2};$$

$$W_{12}(C2) = \int_0^2 ydy + \int_0^1 xdx = \frac{5}{2};$$

$$W_{12}(C3) = \int_0^1 xdx + 2\int_0^1 (2x)dx = \frac{5}{2}$$

For force (b) on the other hand we get

$$W_{12}(C1) = \int_0^1 (0)dx + \int_0^2 (1)dy = 2$$

$$W_{12}(C2) = \int_0^2 (0)dy + \int_0^1 (-2)dx = -2$$

$$W_{12}(C3) = \int_0^1 (-2x)dx + 2\int_0^1 (x)dx = 0$$

Thus we see that whereas the force in (a) gives the work to be the same for all three paths, that in (b) gives different work along the three paths. Thus the first force field may be conservative but the second one is definitely not.

Now let us derive the work-energy theorem in three dimensions. Start from the equation of motion $m\frac{d\vec{v}}{dt} = \vec{F}$ and take the dot product of both sides with the velocity $\vec{v}$ to get

$$m\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{1}{2}m\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \vec{F} \cdot \vec{v}$$

Now integrate both sides with respect to time and use $d\vec{l} = \vec{v}\,dt$, where $d\vec{l}$ is the small distance traveled by the particle in time interval *dt* , to get

$$\frac{1}{2}m\,d(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\,d(v^2) = \vec{F}\cdot\vec{v}dt$$
$$= \vec{F}\cdot d\vec{l}$$

On integration this leads to

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \int_{C(12)} \vec{F}\cdot d\vec{l}$$

This equation tells us that when a force makes a particle move along path C from point 1 to 2, the work done by the force equals the change in its kinetic energy. This is the work-energy theorem in three-dimensions. It is exactly the same as in one dimension except that the work done is calculated by moving along a three-dimensional path.

Potential energy: As is the case in one dimensional motion, potential energy in general can be defined only if the work done is path independent. In that case, the work done depends only on the end points of the path of travel and can be written as the difference on a quantity that is a function of the position vector only. Thus

$$W_{12} = \int_{\vec{r}1}^{\vec{r}2} \vec{F}\cdot d\vec{l} = -U(\vec{r}_2) + U(\vec{r}_1)$$

where $U(\vec{r})$ is defined as the potential energy. Notice that this time I have not written any specific path but just the end points with the integral sign because the work is supposed to be path-independent. From the definition above, it is also evident that here too the difference in the potential energy $U(\vec{r}_2) - U(\vec{r}_1)$ between point *1* and point *2* is the *work done by us* in moving a particle slowly, maintaining its equilibrium, from point *1* to point *2*. Now following the exactly same steps that we did for the one dimensional case, we show that

$$\frac{1}{2}mv_2^2 + U(\vec{r}_2) = \frac{1}{2}mv_1^2 + U(\vec{r}_1)$$

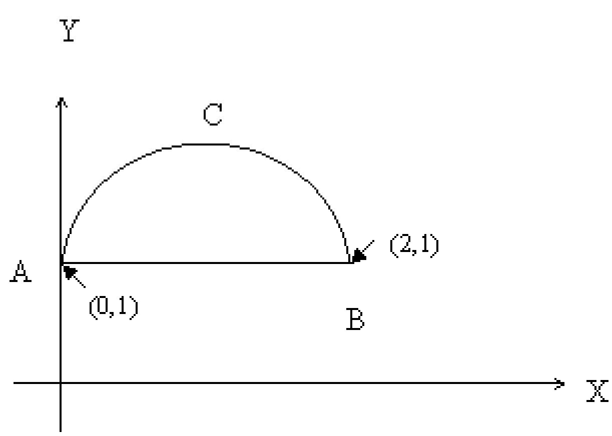
Thus when the potential energy can be defined, the total mechanical *energy of a particle is conserved* . I remind you that the total mechanical energy is the sum of the kinetic and the potential energies. In such cases the force is said to be conservative.

By now you may be wondering how can we find out whether a force is conservative or not. Do we have to calculate the work done along all possible paths before we can say that the force is conservative and therefore the principle of conservation of energy holds good. That certainly would be impossible to do. However, there is a much simpler test to check whether a force field is conservative or not. I am going to tell you about it without giving the proof. To find out about the conservative nature of a force $\vec{F}(\vec{r})$, we calculate its curl $\vec{\nabla} \times \vec{F}(\vec{r})$ defined as

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Now if the curl of a force field vanishes everywhere, it is a conservative force field. On the other hand if the curl of a force field is nonzero, it is not conservative. Let us now apply this test to the two force fields for which we calculated the work done along different paths. For the force field $\vec{F}(x,y) = x\hat{i} + y\hat{j}$, the curl is zero everywhere. Hence it is conservative and, as we saw with three paths, the work done in this field is indeed path independent. On the other hand, for $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$, the curl comes out to be $2\hat{k}$ and therefore the force is not conservative. This was seen above where the work done along the three paths were all different. We now solve an example where knowing the conservative nature beforehand helps us avoid an unnecessary calculation

Example: Take the force field given by $\vec{F}(x,y,z) = A(x^2 y\hat{i} + \dfrac{x^3}{3}\hat{j} + z^3\hat{k})$ and consider a particle moving from A to be along the semicircular path ACB (see figure below). Calculate the difference in its kinetic energy at B and at A.



To calculate the change in the kinetic energy of the particle as it moves from A to B, we should calculate the work done by the force in when the particle travels along the semicircle. For this we should calculate

$$W_{ACB} = \int_{(ACB)} F_x(x,y,z=0)dx + \int_{(ACB)} F_y(x,y,z=0)dy$$

with *y* and *dy* calculated from the equation of the circle $(x-1)^2+(y-1)^2=1$. You should try it and see for yourself that the integrals become really lengthy. On the other hand, if the force is conservative, we can calculate the work done in particle moving along the diameter. The latter calculation is much easier. Let us therefore first calculate the curl of the force. It is

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Ax^2y & A\dfrac{x^3}{3} & Az^3 \end{vmatrix} = 0$$

Thus the work done between any two points is path-independent. We therefore calculate the work along the diameter AB. It is

$$\begin{aligned} W_{AB} &= \int_0^2 F_x(x,y=1,z=0)\,dx \\ &= \int_0^2 Ax^2\,dx \\ &= \frac{4}{3}A \end{aligned}$$

Since the work done is independent of the path, it is going to be the same for the semicircular path ACB also.

After defining the potential energy and getting the principle of conservation energy, we now look a little more at the relationship between the potential energy and the force it gives rise to. As a consequence we also discuss what can we learn about the motion of a particle by looking at its potential energy curve

Learning about force and motion from the potential energy

We learnt above about how the force leads to the concept of potential energy. However, it is the potential energy that is easier to specify than the force. The reason is very simple: force is a vector quantity and as such in specifying it we have to give its three components as a function of position. On the other hand, potential energy is a scalar quantity and is easier to write as a function of position. For the same reason, many a times it is easier to calculate the potential energy than to calculate the force, as we will see in an example below. Thus generally we give the potential energy of a particle to tell about the force field in which the particle is moving. In this section we discuss what can we learn about the motion of a particle by looking at its potential energy.

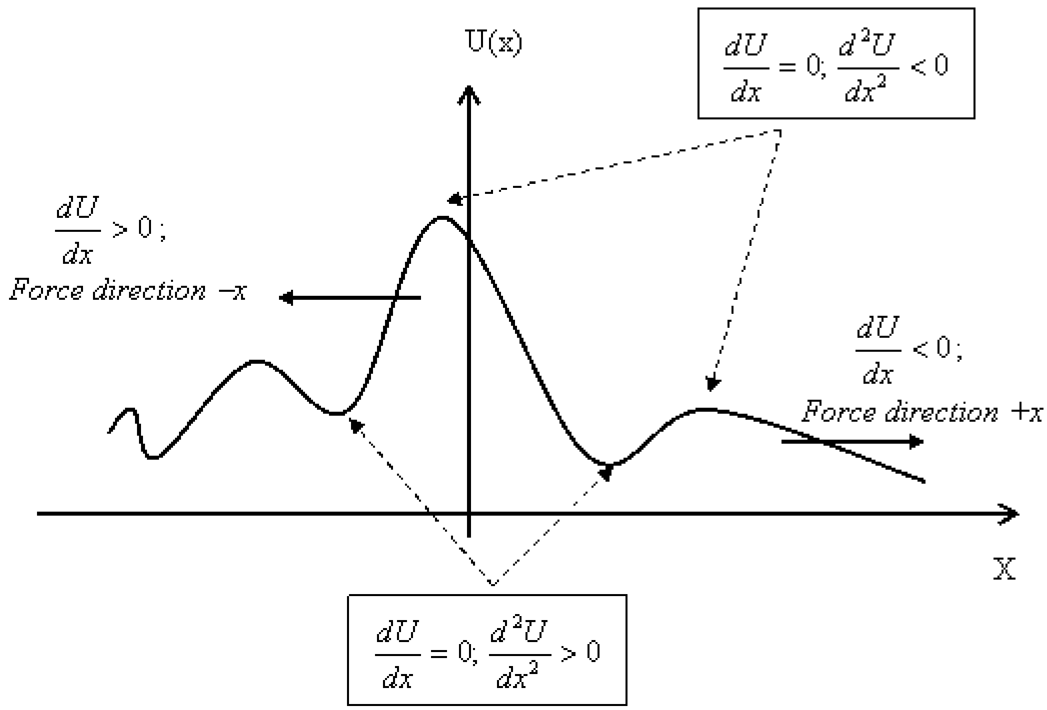
First we discuss how do we get the force from the potential energy. Let us first look at one-dimensional case. Employing the definition of potential energy, we find that for a small displacement Δx

$$F(x)\Delta x = -U(x+\Delta x) + U(x) = -\frac{dU(x)}{dx}\Delta x$$

which means that the force is given by the formula

$$F(x) = -\frac{dU(x)}{dx}$$

This is the key formula relating the force to the potential energy. On the basis of this formula, we can infer a lot about the nature of motion by looking at the potential energy curve. First if $\frac{dU(x)}{dx} > 0$, then the force is towards the negative x-direction and if $\frac{dU(x)}{dx} < 0$, the force is towards the positive x-direction. Thus the force is in the direction of decreasing *U(x)*. What if $\frac{dU(x)}{dx} = 0$? In that case the particle in either on a maximum or a minimum of the potential and there is no force on the particle. The particle is therefore in equilibrium. The equilibrium will be stable one, that is the particle will come back to the equilibrium point when displaced slightly from that point, if it is at the potential energy minimum or equivalently where $\frac{d^2U(x)}{dx^2} > 0$. On the other hand at the maximum of the potential energy, the particle will rush away from that point if it is disturbed. Thus at the potential energy maximum, where $\frac{d^2U(x)}{dx^2} < 0$, the equilibrium is unstable. We see that a particle tends to move towards its potential energy minimum and move away from its potential energy maximum. All these concepts can be shown nicely with a bead moving on a smooth frictionless wire bent in the shape of a curve with many maxima and minima and held in the vertical plane (see the figure below). The potential energy of the bead is then proportional to the height of the curve and as such the wire itself represents the potential energy curve in the figure below.



Now with a bead sliding over the wire, you can easily check that all the points made above about the relationship between the force on the bead and the mathematical properties of the potential energy curve are correct. Further the minima and maxima of the curve are clearly observed to be stable and unstable equilibrium points, respectively.

In three dimensions the equivalent of the derivative is the gradient operator. Thus the force $\vec{F}(\vec{r})$ in two or three dimensions is given as

$$\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r}) = -\left(\frac{\partial U(x,y,z)}{\partial x}\hat{i} + \frac{\partial U(x,y,z)}{\partial y}\hat{j} + \frac{\partial U(x,y,z)}{\partial z}\hat{k}\right)$$

Thus the force is in the direction opposite to that of increasing U. Further, it vanishes wherever the gradient of the potential energy is zero. Individual components of the force are given as

$$F_x = -\frac{\partial U}{\partial x};\quad F_y = -\frac{\partial U}{\partial y}\ \text{ and }\ F_z = -\frac{\partial U}{\partial z}$$

A word of caution is needed here. $\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$ does not mean that if we transform to some other co-ordinates system (say spherical) then

$$\vec{F} = -\frac{\partial U}{\partial r}\hat{r} - \frac{\partial U}{\partial \theta}\hat{\theta} - \frac{\partial U}{\partial \phi}\hat{\phi}$$

will be correct. This is not even dimensionally correct. To get the correct answer, one must properly transform from Cartesian to polar co-ordinates. The result then is

$$\vec{F} = -\frac{\partial U}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

Thus in spherical polar coordinate system, the force components are given as

$$F_r = -\frac{\partial U}{\partial r}; \quad F_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta} \quad and \quad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

Similarly in cylindrical coordinate system the force is related to the potential energy as

$$\vec{F} = -\frac{\partial U}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi} - \frac{\partial U}{\partial z}\hat{z}$$

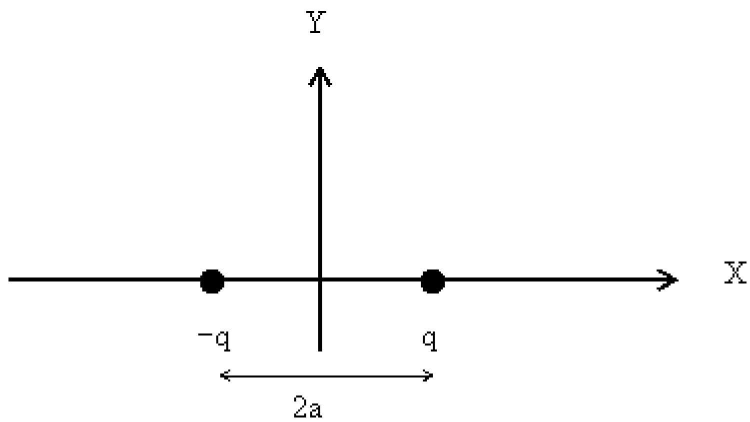
With the individual force components

$$F_r = -\frac{\partial U}{\partial r}; \quad F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi} \quad \text{and} \quad F_z = -\frac{\partial U}{\partial z}$$

Having given you the prescription for obtaining force from the potential energy let us now apply it to find the field of an electric dipole using its scalar potential.

Example: As an application of finding force from the potential, let us calculate the electric field due to a dipole.

Let the dipole be situated at the origin along the x-axis. Let the charges -q and +q be separated by distance $2a$ (see figure below) so that the dipole moment is $\vec{p} = 2qa\hat{i}$. Then potential and field at any point can be calculated by adding the field due to the two charges. Adding the field in this case becomes a bit difficult because we have to obtain three components of the field for each charge and add them. On the other hand, finding the potential is relatively easy because it is a scalar quantity and we obtain it by adding the potential due to two charges. Then the gradient gives the field. In the calculation we assume that $a \rightarrow 0$ and q is correspondingly very large so that their product is finite. We will be using this by keeping term only linear in a and neglecting higher orders.



The potential (potential energy per unit charge) due the two charges is given as

$$U(x,y,z) = \frac{kq}{\sqrt{(x-a)+y^2+z^2}} - \frac{kq}{\sqrt{(x+a)^2+y^2+z^2}}$$

$$= \frac{kq}{\sqrt{x^2+y^2+z^2-2ax}} - \frac{kq}{\sqrt{x^2+y^2+z^2+2ax}}$$

$$= \frac{kq}{r}\left(1+\frac{xa}{r^2}\right) - \frac{kq}{r}\left(1-\frac{xa}{r^2}\right) = \frac{2kqax}{r^3}$$

(Keeping terms up to order a)

$$= \left(\frac{kp}{r^3}x\right)$$

Now taking the gradient we get the three components of the force, which are

$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{kpx}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)$$

$$= -\frac{kp}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} + \frac{3}{2}\frac{2kpx^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}$$

$$= \frac{3kpx^2}{r^5} - \frac{kp}{r^3}$$

Similarly

$$F_y = -\frac{\partial}{\partial y}\left(\frac{kpx}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right) = \frac{3}{2}\left(\frac{2kpxy}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}\right) = \frac{3kpxy}{r^5}$$

$$F_z = -\frac{\partial}{\partial z}\left(\frac{kpx}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right) = \frac{3}{2}\left(\frac{2kpxz}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}\right) = \frac{3kpxz}{r^5}$$

Combining these results together we get for the field of the dipole

$$\vec{E}(\vec{r}) = k\frac{1}{r^3}\left[3(\vec{p}.\hat{r})\hat{r} - \vec{p}\right]$$

I would like you to get the same result by adding the fields of the charges together and compare the answers.

In these lectures, we have learnt: the work-energy theorem, definition of potential and its relationship with the force field, concept of conservative forces and the principle of conservation of energy. I leave these lectures by giving you a few exercises.

Exercise 1: Consider one-dimensional motion in a potential U(x). Show that if a particle of mass *m* is displaced slightly from its equilibrium position at a potential energy minimum at x_0 , it will perform simple harmonic oscillations. Find the corresponding frequency.

Exercise 2: Consider two different inertial frames moving with respect to one another with a constant velocity. Starting from the work-energy theorem in one frame, prove that it is true in the other frame also.

Lecture 17
Collisions

In the previous two lectures, we have seen that when many particles are interacting, there are two conservation systems that are obeyed by them. One, if the net external force on the particles is zero, the total angular momentum of the system remains a constant. This is expressed mathematically as

$$\sum_i \vec{f}_{ext\,i} = 0 \Rightarrow \sum_i m_i\vec{v}_i = \text{constant}$$

$$\text{or } \frac{d}{dt}\left(\sum_i m_i\vec{v}_i\right) = 0$$

Further we saw during the motion of a many particle system, one point - its centre of mass -

moves as if its mass *M* is equal to the total mass $\sum_i m_i$ of the system and the total force $\vec{F}_{ext} = \sum_i \vec{f}_{exti}$ is being applied on that mass. The CM co-ordinate is defines as

$$\vec{R}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$

And it moves according to the equation

$$M\ddot{\vec{R}}_{CM} = \vec{F}_{ext}$$

Thus if $\vec{F}_{ext} = 0$ then $\dot{\vec{R}}_{CM} = \text{constant}$. That means if the total external force on the system is zero, the CM moves with a constant velocity. This is another way of expressing the conservation of linear momentum.

The other conservation principle that we saw was that of total energy. Accordingly the total energy, which is the sum of their kinetic energy KE_i and potential energy PE_i, of a system of particles remains a constant

$$\sum_i \left(KE_i + PE_i\right) = \text{constant}$$

As an example of the power of these principles, in this lecture we apply these two principles to the problem of two particles of masses m_1 and m_2 colliding.

Before we discuss the problem of two particles colliding, we prove something very important and useful: *Kinetic energy of a system of particles is equal to the sum of the kinetic energy of its centre of mass and kinetic energy of particles with respect to the centre of mass* . By kinetic energy of the CM we mean its kinetic energy calculated as a point particle of the total mass $M = \sum_i m_i$ moving with the velocity $\dot{\vec{R}}_{CM}$ of the CM. To see this, substitute in the expression for the kinetic energy

$$\frac{1}{2}\sum_i m_i \vec{v}_i^2$$

$\vec{v}_i = \vec{v}_{CM} + \vec{v}_{ic}$, where $\vec{v}_{CM}$ is the velocity of the CM and $\vec{v}_{ic}$ is the velocity of i^{th} particle in the CM frame. This gives

$$KE = \frac{1}{2}\sum_i m_i(\vec{v}_{CM} + \vec{v}_{ic})^2$$
$$= \frac{1}{2}\left(\sum_i m_i\right)\vec{v}_{CM}^2 + \frac{1}{2}\sum_i m_i\vec{v}_{ic}^2 + \vec{v}_{CM}\cdot\left(\sum_i m_i\vec{v}_{ic}\right)$$

Now $\sum_i m_i\vec{v}_{ic}$ is the momentum of the CM with respect to the CM and therefore proportional to the velocity of the CM with respect to the CM. But the velocity of the CM relative to the CM is zero implying that $\sum_i m_i\vec{v}_{ic} = 0$. This immediately gives

$$KE = \frac{1}{2}M\vec{v}_{CM}^2 + \frac{1}{2}\sum_i m_i\vec{v}_{ic}^2$$

= KE of the CM + KE about the CM

This result, that the kinetic energy of a system of particles can be decomposed into KE of the CM and KE about the CM, is very important and useful. In a later lecture, we will see that the same is true for the angular momentum.

The division of kinetic energy as shown above is useful in learning how energies are shared when particles interact with each-other for short periods of time. As an example take explosion of a bomb. Since the CM will keep on moving the same way as it was before the explosion - because the forces generated are between the pieces of the bomb and therefore have no effect on the total momentum of the system - the explosion does not change the kinetic energy of the CM. Thus all the energy released in the explosion goes to the kinetic energy of the pieces of the bomb with respect to the CM. As another example, consider two particles colliding and getting stuck together. Since the CM keeps on moving with the same speed because of momentum conservation, the minimum kinetic energy that the masses stuck together have to have is that of the centre of mass. Thus the maximum possible energy loss in this case is the sum of their kinetic energy relative to their CM (also called the kinetic energy in the CM frame).

We now get back to the problem of two particles colliding. We consider two particles of masses m_1 and m_2 coming in with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively, interacting in a region, and then going out with velocities $\vec{v}_1'$ and $\vec{v}_2'$ (see figure 1). This is the simplest collision problem. If more particles are involved then the problem is going to be move complicated.

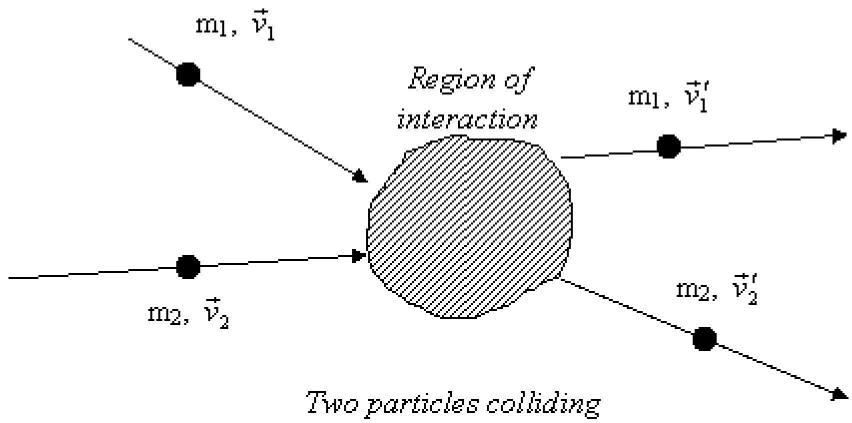


Figure 1

Since we assume particles interact only when they are close to each other, they are essentially free before and after the collision. Further, the interaction region is very small; thus even if the particles are in an external field, the potential energy remains essentially unchanged during the collision. Thus we can write

$$\frac{1}{2}m_1\vec{v}_1^{\,2} + \frac{1}{2}m_2\vec{v}_2^{\,2} + \Delta E = \frac{1}{2}m_1\vec{v}_1'^{\,2} + \frac{1}{2}m_2\vec{v}_2'^{\,2}$$

where we have added ΔE on the left-hand side to take into account any addition or loss of energy during the interaction of particles. For example if the particles generate some energy during interaction, $\Delta E > 0$. This will be the case when two particles release some chemical energy. On the other hand, $\Delta E < 0$ when the particles lose energy during interaction. This is called an inelastic collision. $\Delta E = 0$ is the case of elastic collision; here the total kinetic energy before and after the collision is the same. If particles interact over a large region, we can take the velocities to be in the asymptotic region, where the particles are far apart and therefore the equations above are applicable. The discussion so far has been in terms of balancing the energies involved during the interaction.

The other conservation principle is that of conservation of momentum. Usually during collision the impulse due to collision (internal force if two particles are considered as one system) is much larger than any external impulses. So we neglect it and conserve momentum. If the external impulse comparable to the internal impulse, it must be taken into account. This could be the case when the external force is very large or the particles interact for a long time. For the time being though, we will focus on cases where external impulse can be neglected. Thus

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$$

The two equations are actually a set of four equations with momentum conservation giving three equations, one for each component. However, given $\vec{v}_1, \vec{v}_2$ and ΔE, we have to solve for six quantities, three components for $\vec{v}_1'$ and three for $\vec{v}_2'$. Thus to solve the problem completely, we need more information, for example the scattering angles. In two dimensions also, the conservation equations alone are not enough to solve the problem of finding velocities after the collision. This is because now there will be four unknowns - two components for velocity of each particle - but only three equation, one from the energy balance and two from momentum conservation. Only in one dimension, we can solve the collision problem completely because there are two equations and two unknowns. Nonetheless, we can get a lot of information about the motion from these two conservation laws as we now discuss.

As the first example, let us consider two particles of masses m_1 and m_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively, colliding, getting stuck together to make a particle of mass $(m_1 + m_2)$ that moves with velocity $\vec{V}$. In the process energy ΔE is released. Then moment conservation tells us

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{V}$$

and balancing the energy gives

$$\frac{1}{2}m_1\vec{v}_1^{\,2} + \frac{1}{2}m_2\vec{v}_2^{\,2} + \Delta E = \frac{1}{2}(m_1 + m_2)\vec{V}^2$$

Notice that we have added to ΔE to the left-hand side so that the total final kinetic energy is the sum of the total initial kinetic energy and the energy added to the system. Substituting for $\vec{V}$ from the momentum conservation equation in the energy equation, we get

$$m_1\vec{v}_1^{\,2} + m_2\vec{v}_2^{\,2} + 2\Delta E = \frac{(m_1 + m_2)(m_1\vec{v}_1 + m_2\vec{v}_2)^2}{(m_1 + m_2)^2}$$

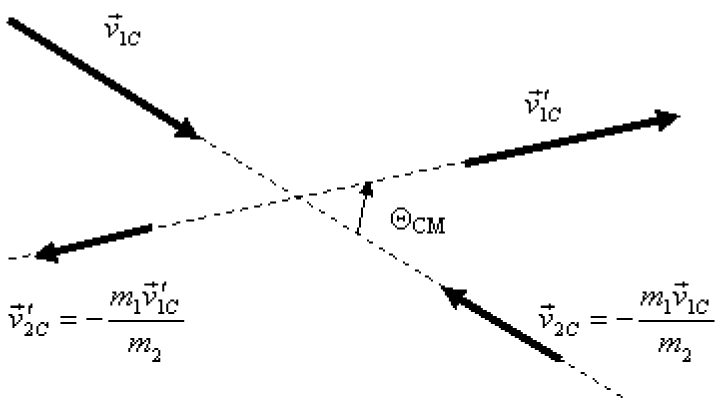
which on simplification gives

$$(\vec{v}_1 - \vec{v}_2)^2 = -\frac{2\Delta E(m_1 + m_2)}{(m_1 + m_2)}$$

The left-hand side of the equation above is definitely positive. On the other hand, the right-hand side is negative if $\Delta E > 0$, i.e., the final kinetic energy is larger than the initial kinetic energy. So this reaction will not be possible if it is exothermic, i.e., some energy is generated and added to the initial kinetic energy. Thus two atoms colliding in free space will not combine to form a molecule (in which process the energy is usually released). However if energy is taken away from the system, i.e. $\Delta E < 0$, then the reaction is possible. This is the information we have got purely on the basis of conservation laws. We now go on to discuss collisions as described with respect to the CM. We will see that this gives us a lot of insight into the collision problem.

As we had stated earlier, the conservation of momentum implies that the centre of mass moves with a constant velocity when there is no external force on the particles. Thus if we attach a frame to the CM, it will also move with constant velocity and will be an initial frame of reference. Let us call this the CM frame. Since it is an inertial frame, we can equally well describe a collision process is a CM frame. Observing a collision from the CM frame gives us the biggest advantage that the sum of the momenta (the total momentum) is always zero in this frame. In this lecture we will be focusing on two particle collisions as described from the CM frame. We will see that because of the total momentum being zero, description of a collision in this frame becomes simpler. In coming lectures we will see that CM provides a convenient origin for studying rotational motion also.

For now, let us look at the two particles collision. As stated above, in the CM frame the total momentum is always zero because in this frame the CM does not move. So that the velocities of two particles in the CM frame are always in the direction opposite to each other. Further the motion remains confined to a plane formed by the lines representing the initial and the final velocities directions (keep in mind that the velocities of the two particles at any instant are along the same line though opposite in direction). Thus in the CM frame a collision looks as shown in figure 2.



A two particle collision observed from the CM frame. Incoming velocities are shown by unprimed symbols whereas outgoing velocities are primed. Scattering angle is Θ_{CM}.

Figure 2

In figure 2 two particles with masses m_1 and m_2 and velocities $\vec{v}_{1C}$ and $\vec{v}_{2C} = -\frac{m_1\vec{v}_{1C}}{m_2}$ are coming in for a collision; they collide and particle *1* goes out with velocity $\vec{v}'_{1C}$ and particle 2 with $\vec{v}'_{2C} = -\frac{m_1\vec{v}'_{1C}}{m_2}$. In the process particle *1* gets deflected by an angle Θ_{CM}. As stated earlier, even in 2d there are four unknowns: two components of $\vec{v}'_{1C}$ and two of $\vec{v}'_{2C}$ to be obtained but only three equations- one for energy conservation and two for momentum conservation. So the problem cannot be solved fully by using conservation principle only. However, if the interaction is known, then Θ_{CM} and both the velocities after collision can in principle be calculated. Let us now see how much can we learn about the motion after collision applying only the conservation principles. We will be discussing both the elastic and inelastic collisions. Recall that if the kinetic energy remains unchanged in a collision, the collision is elastic; on the other hand, if the energy is lost the collision is inelastic.

Let us first focus on an elastic collision and analyze it in the CM frame. As pointed out earlier, the velocities of the two particles before and after collision are opposite to each other. Thus the relationship between the magnitudes v_{1C} , v_{2C} , v'_{1C} and v'_{2C} of the velocities is

$$m_1 v_{1C} = m_2 v_{2C}$$

$$m_1 v'_{1C} = m_2 v'_{2C}$$

$$\frac{1}{2} m_1 v_{1C}^2 + \frac{1}{2} m_2 v_{2C}^2 = \frac{1}{2} m_1 v_{1C}'^2 + \frac{1}{2} m_2 v_{2C}'^2$$

Substituting for v_{2C} and v'_{2C} from the first two equations in the last one we get

$$v'_{1C} = v_{1C} \quad \text{and} \quad v'_{2C} = v_{2C}$$

Thus the velocity vectors of both particles just rotate but do not change in magnitude as the partial move out after collision. You have learnt in previous classes that in an elastic collision the magnitude of the relative velocity of one particle with respect to the other remains unchanged during the collision. In one dimension it means that the speed of approach of two particles is the same as their speed of separation. Let us now see how it follows directly from the conservation principles.

As we have derived above, $v'_{1C} = v_{1C}$ and $v'_{2C} = v_{2C}$ in an elastic collision. If the velocities of the two particles are $\vec{v}_1$ and $\vec{v}_2$, respectively, in the ground frame, then

$$\vec{v}_1 = \vec{v}_{CM} + \vec{v}_{1C} \text{ and } \vec{v}_2 = \vec{v}_{CM} + \vec{v}_{2C}$$

Similar relationships hold for the velocities after collision i.e.

$$\vec{v}'_1 = \vec{v}_{CM} + \vec{v}'_{1C} \text{ and } \vec{v}'_2 = \vec{v}_{CM} + \vec{v}'_{2C}$$

Using these relationships we find that

$$
\begin{aligned}
(\vec{v}_2 - \vec{v}_1)^2 &= (\vec{v}_{2C} - \vec{v}_{1C})^2 \\
&= v_{2C}^2 + v_{1C}^2 - 2\vec{v}_{2C} \cdot \vec{v}_{1C} \\
&= v_{2C}^2 + v_{1C}^2 + 2v_{2C} \cdot v_{1C} \\
&\quad \left(\text{because } \vec{v}_{2C} = -\vec{v}_{1C}\right)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(\vec{v}_2' - \vec{v}_1')^2 &= (\vec{v}_{2C}' - \vec{v}_{1C}')^2 \\
&= v_{2C}'^2 + v_{1C}'^2 - 2\vec{v}_{2C}' \cdot \vec{v}_{1C}' \\
&= v_{2C}^2 + v_{1C}^2 + 2v_{2C} \cdot v_{1C}
\end{aligned}
$$

$$\left(\text{because } v_{1C}' = v_{1C},\ v_{2C}' = v_{2C} \text{ and } \vec{v}_{2C}' = -\vec{v}_{1C}'\right)$$

Thus we see that in an elastic collision

$$(\vec{v}_2' - \vec{v}_1')^2 = (\vec{v}_2 - \vec{v}_1)^2$$

We have shown that the magnitude of relative velocity of one particle with respect to other remains the same in an elastic collision.

To see the dramatic effects of a nearly elastic collision, take a table-tennis ball (very small mass *m*), put it on a large bouncy ball of mass *M (M >> m)* , and drop them from a height (see figure 4) on a hard floor. You will see that the table-tennis ball bounces back really high after the balls hit the ground. Can you work how high will it go if the balls are dropped from a height h? Assume that no energy is lost.

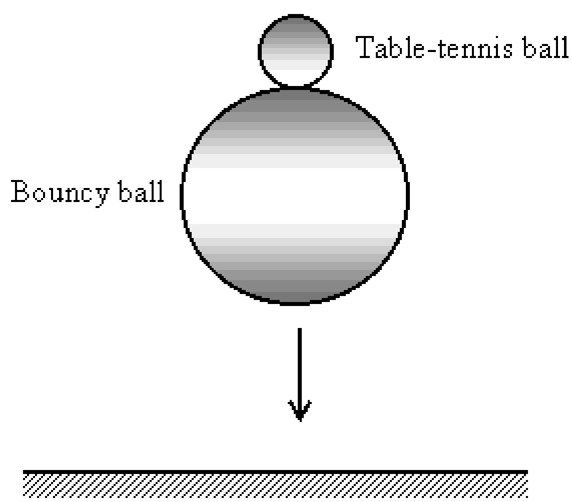
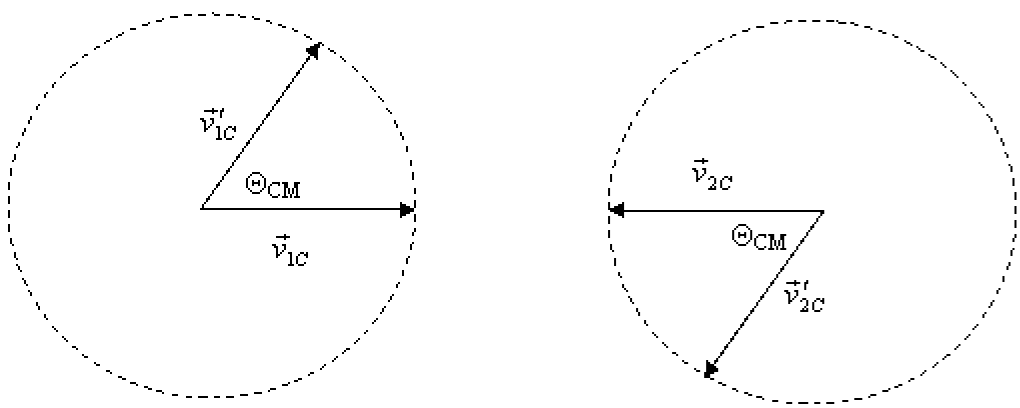


Figure 4

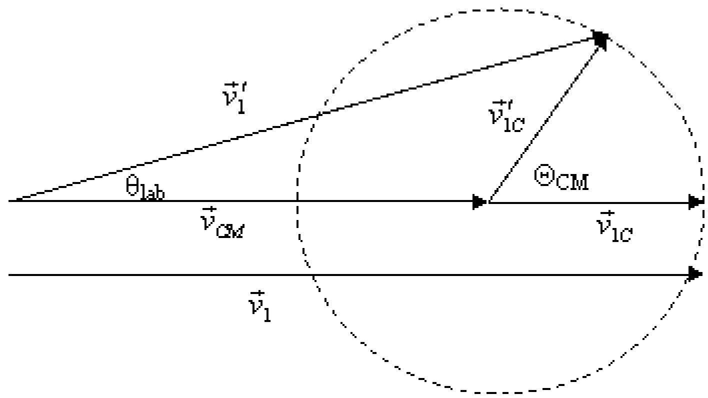
Now we consider a two-particle elastic collision in a plane and analyze it. This could be the collision of a striker and a coin on a carom board, for example. It is a two-dimensional case. We are going to analyze the motion graphically. First we look at the velocities in the CM frame. If we take the initial direction of particle 1 towards $+x$, the velocities of the two particles before and after collision can be shown as done in figure 5. Keep in mind that in an elastic collision, the magnitude of the velocities of each particle remains unchanged in the CM frame. However the direction of the velocity for each particle changes by an angle Θ_{CM}. as shown in figure 5.



Velocities of two colliding particles before and after collision as seen in the CM frame

Figure 5

The picture above shows the angle of scattering in the CM frame. However experiments are done on ground - and not in the CM frame. So we should be answering the question: by what angle θ_{lab} does particle 1 scatter in the laboratory frame? Since velocities $\vec{v}_1$ and $\vec{v}'_1$ in the lab frame are given as $\vec{v}_1 = \vec{v}_{CM} + \vec{v}_{1C}$ and $\vec{v}_2 = \vec{v}_{CM} + \vec{v}_{2C}$, the relationship between these velocities can be shown as done in figure 6.



Velocities of particle 1 before and after collision as seen in the CM and the lab frame.

Figure 6

From figure 6, it is now very easy to see that

$$\tan\theta_{lab} = \frac{v_{1C}'\sin\Theta_{CM}}{v_{CM} + v_{1C}'\cos\Theta_{CM}} = \frac{v_{1C}\sin\Theta_{CM}}{v_{CM} + v_{1C}\cos\Theta_{CM}} = \frac{\sin\Theta_{CM}}{(v_{CM}/v_{1C}) + \cos\Theta_{CM}}$$

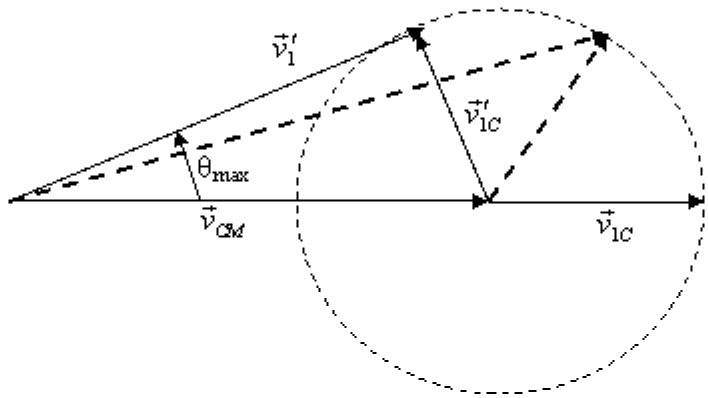
Similar relationships can also be derived for particle 2. Now if particle 2 was at rest when hit by particle 1, then

$$v_{CM} = \frac{m_1 v_1}{m_1 + m_2} \quad \text{and} \quad v_{1C} = \frac{m_2 v_1}{m_1 + m_2}$$

This gives

$$\tan\theta_{lab} = \frac{\sin\Theta_{CM}}{(m_1/m_2) + \cos\Theta_{CM}}$$

Let us now look at two cases: $m_1 > m_2$ and $m_1 < m_2$. In the case of $m_1 > m_2$, θ_{lab} cannot be greater than a particular angle θ_{max}. This can be either calculated by using the expression above or alternatively, graphically as we do. For $m_1 > m_2$ we also have $v_{CM} > v_{1C}$. Thus a picture showing the velocities in the laboratory and the CM frame looks like that in figure 7.



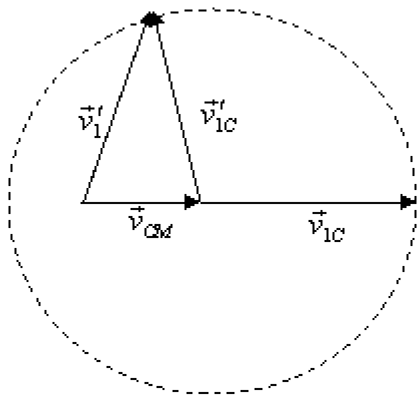
Deflection angle of particle 1 in the lab frame is maximum when velocities $\vec{v}_1'$ and $\vec{v}_{1C}'$ are perpendicular.

Figure 7

It is clear from figure 7 that the deflection angle of particle 1, when hitting another particle of smaller mass, increases as Θ_{CM} increases from zero. It is maximum when the velocities $\vec{v}_1'$ and $\vec{v}_{1C}'$ are perpendicular. If $\vec{v}_{1C}'$ is rotates beyond this angle, deflection starts becoming smaller. Thus θ_{max} is given by the formula

$$\sin\theta_{max} = \frac{v_{1C}'}{v_{CM}} = \frac{v_{1C}}{v_{CM}} = \frac{m_2}{m_1}$$

It is clear from the expression above that when a particle hits a lighter particle at rest, it is deflected by a small angle. This is reasonable as a light particle can hardy deflect a heavier particle. Thus the heavier particle keeps on moving forward even after the collision. On the other hand, there is no restriction on the scattering angle when a light particle hits a heavier particle at rest i.e. $m_1 < m_2$. In this case $v_{CM} < v_{1C}$ and therefore the graphical representation of different velocities is as shown in figure 8.



Deflection angle of particle 1 in the lab can take any value when $m_1<m_2$.

Figure 8

It is clear from the figure that as Θ_{CM} increases, so does θ_{lab}. In this situation, however, there is no restriction on the value that θ_{lab} can take as Θ_{CM} sweeps angles from 0 to 2π .

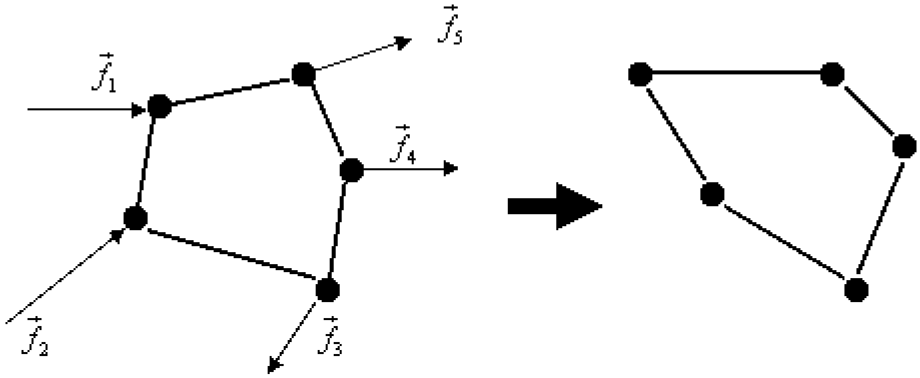
So far we have focused on elastic collisions only and could learn a great deal about them from conservation laws for momentum and energy. Such general conclusions are difficult to draw for inelastic collisions. As discussed in the beginning of this lecture, for inelastic collisions, we can definitely say that the maximum possible loss of energy is equal to the kinetic energy of particles in their CM frame. This would occur when the colliding particles get stuck together so that their kinetic energy after collision is zero in the CM frame. This concludes our lecture on collisions as analyzed using conservation laws.

Lecture 18
Rotational dynamics I: Angular momentum

So far we have applied Newton's laws to point particles and the CM motion for a collection of particles. We are now going to look at what happens beyond the motion of the CM, which is described by the equation

$$\vec{F}_{total} = M_{total}\frac{d^2\vec{R}_{CM}}{dt^2}$$

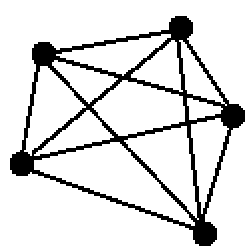
Let us see what else could happen to a body made up of a collection of particles where forces are applied at each point (figure 1). The particles are connected with flexible attachments shown as lines.



Distortion of a collection of particles connected with each other moving under force on each particle.

Figure 1

In the figure above, although the CM moves with $\ddot{\vec{R}}_{CM} = \dfrac{\vec{F}_{Net}}{M}$, the body itself could deform and change its orientation. Thus the distances between the particles and the angles between lines joining them would change. This is the most general motion that could take place. In the next few lectures we want to focus only one of the effects of the force applied. We are going to assume that a body only changes its orientation but does not deform. This is achieved by keeping the distance between any two particles of the body unchanged. Such a body is known as a *rigid body* . Thus in the example above, if we connect all the particles with each other by rods of fixed length, the body will become rigid. This is shown in figure 2.

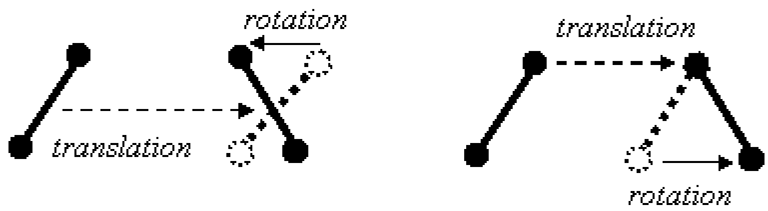


A body is rigid if the distance between any of its two particles remains unchanged.

Figure 2

The only possible motion of such a body is a translation plus a change in its orientation. The simplest example of a rigid body is two masses attached at the ends of a rod of fixed length. On the other hand, a tin-can partially filled with sand is not a rigid body since the distance between two particles keeps on changing with the motion of the can.

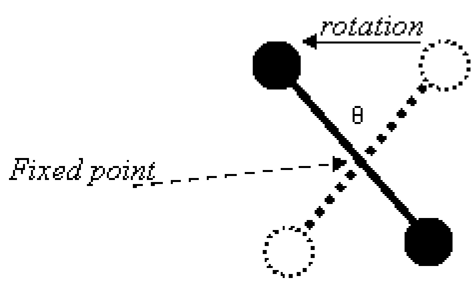
As stated above, the most general motion of a rigid body is its translation plus its change of orientation. The latter is equivalent to a rotation about a point. The beauty of this decomposition is that to get the final position of the body, we can translate any point in the body and then rotate the body about that point. Irrespective of which point we choose, the sense and the angle of rotation is always the same. Usually this point is taken to be the CM for reasons that will become clear later lectures. This general motion is shown below in figure 3, giving two possible ways of translating and rotating the body.



The most general motion of a rigid body is its translation plus a rotation. Two such possible ways giving the same final orientation are shown. In both cases the sense and angle of rotation are the same

Figure 3

You see that in figure 3 the rigid body has translated and also rotated. On the other hand, if we keep one of the points on the body fixed the only thing the body can do is to change its orientation (see figure 4). Thus with a point fixed, the only possible motion of a rigid body is a rotation.



The only possible motion of a rigid body with one of its points fixed is a rotation.

Figure 4

A question that arises now is how many variables do we need to specify the general motion of a rigid body. It requires three variables - x, y and z coordinates of the point that is translated - to describe the translation, and three more - angle of rotation about each axis - to represent the rotation. You can see that in general a rigid body would require six variables to describe its motion. However, if one of its points is fixed, three variables are sufficient to specify its rotation. So we conclude *a rigid body needs six parameters to describe its motion*.

For simplicity, in the beginning we are going to focus on rigid moving with its one point fixed. Thus it will change only by changing its orientation. We will further simplify the problem by considering rotation about an axis fixed in space. In the next step, we will allow the axis to translate but without changing its orientation. Finally we will also let the orientation of the axis change. Thus we will increase the complexity of the problem gradually.

Dynamics of rigid body: The dynamics of a rigid body is best described by considering its angular momentum. You can think of angular momentum as the rotational counter part of linear momentum. This quantity is central to describing rotational motion of a rigid body. So let us first spend some time in understanding this quantity. Although we are introducing angular momentum here in the context of rigid bodies, the treatment below is quite general.

For a single particle moving with linear momentum $\vec{p}$ at a distance $\vec{r}$ from the origin the angular momentum $\vec{L}$ is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

You can immediately see that it is an origin-dependent quantity. If we calculate it with respect to some other point, it will come out to be different. If a particle of mass *m* is moving in a plane then using the polar coordinates for it, it is easily shown that its angular momentum is $\vec{L} = mr^2\dot{\phi}\hat{z}$. Let us now find out what is the rate of change of angular moment? It is calculated below.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$
$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

With $\frac{d\vec{r}}{dt} = \vec{v}$, $\vec{p} = m\vec{v}$ and $\frac{d\vec{p}}{dt} = \vec{F}$, where $\vec{F}$ is the force on the particle, the equation above is simplified to

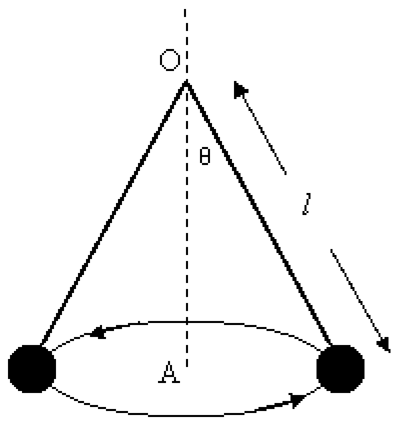
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}\,(torque)$$

Thus rate of change of angular momentum is equal to the torque applied on the body. From the equation above, the law of *conservation of angular momentum* follows immediately: If the applied torque $\vec{\tau} = 0$ the angular momentum $\vec{L}$ does not change, i.e. it is a constant. The equation

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

is the angular momentum equivalent of Newton 's IInd law. Let us now illustrate the ideas presented so far with the example of a conical pendulum.

Example 1: A conical pendulum is like the regular pendulum with a light (mass $m = 0$) rigid rod carrying a bob of mass *m* at one of its ends. The other end is fixed and the bob moves in a circle with speed *v* (see figure 6). We wish to calculate the tension in the rod and the angle θ it makes from the vertical by applying the angular momentum-torque equation.



A conical pendulum

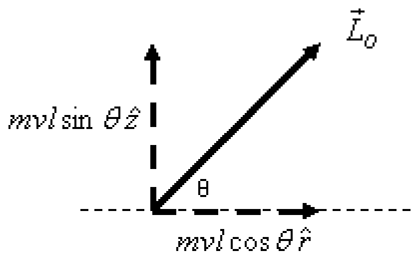
Figure 6

Let us first calculate angular momentum about point O . We will use cylindrical co-ordinates because of the symmetry of the problem. With respect to O

$$\vec{r} = l\sin\theta\,\hat{r} - l\cos\theta\,\hat{z} \quad \text{and} \quad \vec{v} = v\hat{\phi}$$

$$\begin{aligned}\vec{L}_O &= m\vec{r}\times\vec{v}\\ &= m(l\sin\theta\,\hat{r} - l\cos\theta\,\hat{z})\times v\hat{\phi}\\ &= mvl\sin\theta\,\hat{z} + mvl\cos\theta\,\hat{r}\end{aligned}$$

The $\vec{L}_O$ vector looks as shown in figure 7, when the bob of the pendulum is in the paper plane.



The angular momentum and its components for a conical pendulum

Figure 7

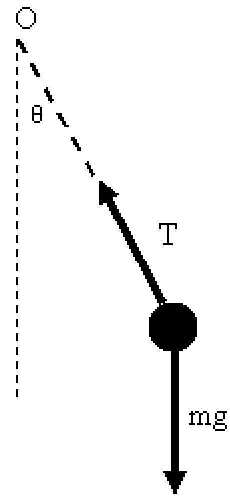
So the angular momentum $\vec{L}$ is perpendicular to the rod (take the dot product with $\vec{r} = l\sin\,\theta\hat{r} - l\cos\theta\hat{z}$ for mass *m* and see for yourself) and as the particle rotates the horizontal component of $\vec{L}$ are rotates with it and the vertical component remains a constant. Let us now apply the equation

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_O$$

We have

$$\begin{aligned}\frac{d\vec{L}_O}{dt} &= \frac{d}{dt}\left(mvl\sin\,\theta\hat{z} + mvl\cos\theta\hat{r}\right)\\ &= mvl\cos\theta\frac{d}{dt}\hat{r}\\ &= mvl\cos\theta\,\dot{\phi}\hat{\phi}\\ &= mvl\cos\theta\times\frac{v}{l\sin\,\theta}\hat{\phi}\\ &= mv^2\cot\theta\hat{\phi}\end{aligned}$$

We now calculate the torque acting on the pendulum. There are two forces, the tension $\vec{T}$ and the weight $W = -mg\hat{k}$, acting on the particle as shown in figure 8.



Forces acting on the mass m of the conical pendulum

Figure 8

But $\vec{T}$ passes through *O* and does not give any torque. Thus

$$\vec{\tau}_O = (l\sin\theta\hat{r} - l\cos\theta\hat{z}) \times mg(-\hat{z})$$
$$= mg\,l\sin\theta\hat{\phi}$$

Substituting these in the angular momentum-torque equation $\frac{d\vec{L}_O}{dt} = \vec{\tau}_O$ then gives

$$\frac{v^2}{gl} = \tan\theta\sin\theta$$

The angular momentum-torque equation therefore gives us the angle θ that the pendulum makes with the vertical. How do we find the tension T ? On the other hand, applying Newton 's second Law we get

$$T\cos\theta = mg \quad \text{and} \quad T\sin\theta = \frac{mv^2}{l\sin\theta}$$

giving

$$T = \frac{mg}{\cos\theta} \quad \textit{and} \quad \tan\theta\sin\theta = \frac{v^2}{gl}$$

These equations give us both T and θ, but the equation $\frac{d\vec{L}_O}{dt} = \vec{\tau}_O$ gives only the angular relationship. Does this mean that the angular-momentum torque equation is not equivalent to Newton 's second law? The answer is that it is. It so happens that in applying the equation about O, when cross products $\vec{r} \times \vec{p}$ and $\vec{r} \times \vec{F}$ are taken, some components of the force do not contribute to the torque and drop out of the equation. For example in this case $\vec{r} \times \vec{T}$ becomes zero. To get full solution, therefore, we now apply $\frac{d\vec{L}}{dt} = \vec{\tau}$ about point A. Taking A as the origin we have

$$\vec{r} = l\sin\theta\,\hat{r} \quad \text{and} \quad \vec{v} = v\hat{\phi}$$
$$\vec{L}_A = ml\sin\theta\,\hat{r} \times v\hat{\phi}$$
$$= mvl\sin\theta\,\hat{z}$$

Since all the quantities in $\vec{L}_A$ are constants, we have

$$\vec{\tau}_A = \frac{d\vec{L}_A}{dt} = 0$$

Let us calculate the torque $\vec{\tau}_A$ about A. With A as the origin, the forces are given as

$$\vec{T} = -T\sin\theta\,\hat{r} + T\cos\theta\,\hat{z} \quad \text{and} \quad m\vec{g} = -mg\hat{z}$$

Therefore

$$\vec{\tau}_A = l\sin\theta\,\hat{r} \times \{-T\sin\theta\,\hat{r} + (T\cos\theta - mg)\}\hat{z} = 0$$
$$\Rightarrow \quad (T\cos\theta - mg)l\sin\theta\,\hat{\phi} = 0$$

which gives

$$T = \frac{mg}{\cos\theta}$$

Thus applying $\frac{d\vec{L}}{dt} = \vec{\tau}$ about two different points gives exactly the same solution as that obtained from $\frac{d\vec{p}}{dt} = \vec{F}$. Thus the two ways of solving the problem are equivalent. Through this example I have shown you (a) *the origin dependence of* $\vec{L}$ and $\vec{\tau}$, and (b) *equivalence of*

$$\frac{d\vec{L}}{dt} = \vec{\tau} \quad \textit{and} \quad \frac{d\vec{p}}{dt} = \vec{F} \; .$$

Let me now illustrate conservation of angular momentum by a well known example: that of Kepler's Law of equal area concept in equal time. Accordingly, when planets are going around the sun, the rate at which their position vector from the sun sweeps the area is a constant. Recall from the lecture on polar coordinates that for a particle moving under a radial force, we had obtained that $r^2\dot{\phi}$ is a constant. This is nothing but two times the rate of area sweep by the radius vector. We now want to get this law from the conservation of angular momentum.

For a planet, we know that the force is in redial direction. So that the torque

$$\vec{\tau} = \vec{r} \times \vec{F} = 0$$

Thus

$$\frac{d\vec{L}}{dt} = \vec{\tau} = 0 \;\; \text{or} \;\; \vec{L} = \text{constant}$$

Since $\vec{L} = mr\hat{r} \times \vec{v} = mr\hat{r} \times (\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) = mr^2\dot{\phi}\hat{z}$, its constancy means

$$\mathrm{r}^2\dot{\phi} = \text{constant}$$

which is Kepler's second law.

After this initial demonstration of $\dfrac{d\vec{L}}{dt} = \vec{\tau}$ with a single particle, we move on to a system of many particles. It is really a system of many particles that we are dealing with in rigid-body dynamics.

Angular Momentum of a collection of particles: If there are many particles then the total angular momentum $\vec{L}$ about a point *O* is the sum of individual angular momenta of each particle about *O* . Thus

$$\begin{aligned} \vec{L} &= \sum_i \vec{L}_i \\ &= \sum_i m_i \vec{r}_i \times \vec{p}_i \end{aligned}$$

As for the angular momentum of a single particle, the angular momentum of a many-particle system is also origin-dependent. (***Question:*** *Under what conditions will the angular momentum be independent of the origin?)*

Now recall that the kinetic energy for a collection of particles is the sum of the kinetic energy of their centre of mass (CM) and the kinetic energy of particles with respect to the CM. Interestingly the angular momentum of a many-particle system can be expressed in the same manner. Thus the total angular momentum of a collection of particles is equal to the angular momentum of the CM plus the angular momentum of particles about the CM. Let us now prove it. To do so express the position vector and the velocity of a particle as

$$\vec{r}_i = \vec{r}_{CM} + \vec{r}_{iC} \quad \text{and} \quad \vec{v}_i = \vec{v}_{CM} + \vec{v}_{iC}$$

where $\vec{r}_{CM}$ and $\vec{v}_{CM}$ refer to the position and velocity of the CM and $\vec{r}_{iC}$ and $\vec{v}_{iC}$ the position and velocity of ith particle with respect to the CM. Now the total angular momentum can be writes as

$$\begin{aligned}\vec{L} &= \sum_i m_i(\vec{r}_{CM} + \vec{r}_{iC}) \times (\vec{v}_{CM} + \vec{v}_{iC}) \\ &= \left(\sum_i m_i\right)\vec{r}_{CM} \times \vec{v}_{CM} + \vec{r}_{CM} \times \left(\sum_i m_i \vec{v}_{iC}\right) \\ &\quad + \left(\sum_i m_i \vec{r}_{iC}\right) \times \vec{v}_{CM} + \sum_i m_i \vec{r}_{iC} \times \vec{v}_{iC}\end{aligned}$$

However, by definition of the CM, $\left(\sum_i m_i \vec{r}_{iC}\right) = 0$ and $\left(\sum_i m_i \vec{v}_{iC}\right) = 0$. Therefore the second and the last term in the expression above do not contribute. The remaining terms are written as

$$\begin{aligned}\vec{L} &= \left(\sum_i m_i\right)\vec{r}_{CM} \times \vec{v}_{CM} + \sum_i m_i \vec{r}_{iC} \times \vec{v}_{iC} \\ &= M\vec{r}_{CM} \times \vec{v}_{CM} + \sum_i m_i \vec{r}_{iC} \times \vec{v}_{iC} \\ &= \vec{L}_{CM} + \vec{L}_{aboutCM}\end{aligned}$$

where M is the total mass of the system. This is a remarkable result, and as we will see, facilitates calculations involving rigid-body dynamics a lot. Keep in mind though that this result is true only with for the CM. For an arbitrary point O' in the body, we cannot write

$$\vec{L} = \vec{L}_{o'} + \vec{L}_{about\,o'}$$

because $\vec{L} = \vec{L}_{CM} + \vec{L}_{about\,CM}$ depends explicitly on the definition of the CM. We will later use this fact to obtain the parallel axis theorem that you may have learnt in your previous classes. The theorem is similar to the transfer theorem of the second-moment of an area.

The relationship $\vec{L} = \vec{L}_{CM} + \vec{L}_{about\,CM}$ also tells us that if the total momentum of a system of particles is zero, its angular momentum will be independent of the origin. I leave the simple proof for you to work out.

Example: Take a bicycle wheel of radius *R* rolling along the ground and assume all its mass *M* is concentrated along the rim. If it is rolling without slipping then its motion is as follows: its CM moves with speed *V* along a straight line and the wheel rotates about the CM with angular speed $\omega = \frac{V}{R}$ so that the point on ground is at rest. We want to find its angular momentum in a frame stuck to the ground such that the wheel is moving along its x-axis see figure 9).

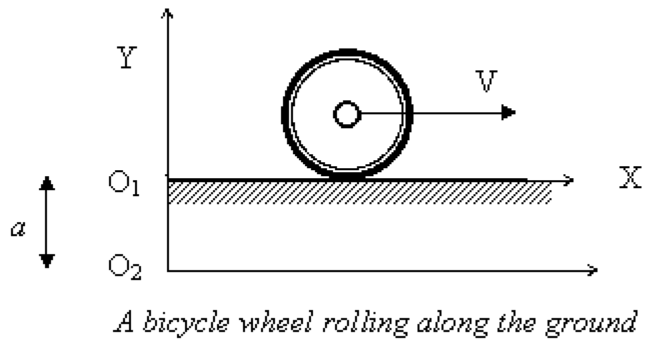


Figure 9

The angular momentum of the wheel about its CM is given as

$$\begin{aligned}\vec{L}_{about\,CM} &= \left(\sum_i m_i\right) RV \\ &= MRV\end{aligned}$$

So angular momentum about the origin O_1 (see figure 9) would be

$$\begin{aligned}&= MRV + angular\ momentum\ of\ CM\ about\ O_1 \\ &= MRV + MRV \\ &= 2MRV\end{aligned}$$

On the other hand, if we were to calculate the angular momentum about O_2 (see figure 9) it would come out to be

$$\begin{aligned}&= MRV + angular\ momentum\ of\ CM\ about\ O_2 \\ &= MRV + M(R+a)V \\ &= 2MRV + MRa\end{aligned}$$

Notice that in both the cases we have added the angular momentum of the CM and that about the CM. It is because their directions come out to be the same (negative z direction). One must be careful about these things because angular momentum is a vector quantity. Having introduced you to the concept of angular momentum, I now discuss about the rate of its change for a many-particle system where the particles are interacting with each other also.

Dynamic of a rigid body; $\frac{d\vec{L}}{dt}$ and conservation of angular momentum: Let us now look at $\frac{d\vec{L}}{dt}$ in the case of a collection of particles which are interacting with each other and are also being acted upon by external forces.

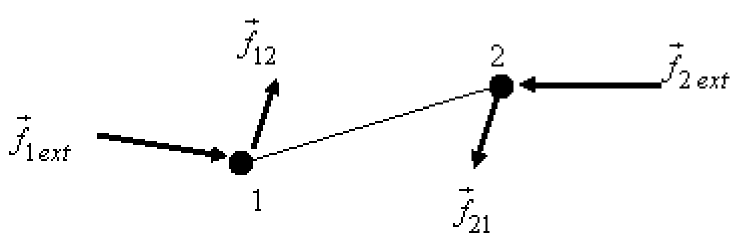
$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_i m_i \vec{r}_i \times \vec{v}_i$$

$$= \sum_i m_i \vec{r}_i \times \frac{d\vec{v}_i}{dt}$$

But $m_i \frac{d\vec{v}_i}{dt} = \vec{f}_i$ ($\vec{f}_i$ s the total force, i.e. the sum of external and internal forces on the particle). This gives

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{f}_i$$

Before simplifying this equation in terms of the external torque, let us see where does this equation lead us for a two particle system shown in figure 10?



A two particle system with particles interacting with each other and are also being acted upon by external forces

Figure 10

The two particles 1 and 2 shown in figure 10 are external forces $\vec{f}_{1ext}$ and $\vec{f}_{2ext}$, respectively. They also interact with each other with particle 2 applying a force $\vec{f}_{12}$ on particle 1 and particle 1 applying a force $\vec{f}_{21}$ on particle 2. We assume the forces to be following Newton 's IIIrd law so that $\vec{f}_{21} = -\vec{f}_{12}$. Now the rate of change for this system can be written as

$$\begin{aligned}
\frac{d\vec{L}}{dt} &= \vec{r}_1 \times (\vec{f}_{1ext} + \vec{f}_{12}) + \vec{r}_2 \times (\vec{f}_{2ext} + \vec{f}_{21}) \\
&= \vec{r}_1 \times \vec{f}_{1ext} + \vec{r}_2 \times \vec{f}_{2ext} + \vec{r}_1 \times \vec{f}_{12} + \vec{r}_2 \times \vec{f}_{21} \\
&= \vec{r}_1 \times \vec{f}_{1ext} + \vec{r}_2 \times \vec{f}_{2ext} + (\vec{r}_1 - \vec{r}_2) \times \vec{f}_{12} \\
&= \vec{\tau}_{ext} + (\vec{r}_1 - \vec{r}_2) \times \vec{f}_{12}
\end{aligned}$$

Thus the rate of change of angular momentum is equal to only the external torque if $(\vec{r}_1 - \vec{r}_2) \times \vec{f}_{12} = 0$ or $\vec{f}_{12} \parallel (\vec{r}_1 - \vec{r}_2)$,i.e. the force between the particles is along the line joining them. At this point I would like you to recall that in the case of linear momentum, the rate of change on linear momentum equals the total external force, i.e. $\frac{d\vec{p}}{dt} = \vec{F}_{ext}$, only if $\vec{f}_{12} = -\vec{f}_{21}$. For angular momentum to satisfy $\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$, the additional condition of $\vec{f}_{12} \parallel (\vec{r}_1 - \vec{r}_2)$ is also needed. Fortunately for most of the mechanical applications this is true. Let us now generalize this to the case of a many-particle system. For such a system

$$\begin{aligned}
\frac{d\vec{L}}{dt} &= \sum \vec{r}_i \times \vec{f}_i \\
&= \sum_i \vec{r}_i \times \vec{f}_{ext} + \sum_i \sum_{j, j \neq i} \vec{r}_i \times \vec{f}_{ij}
\end{aligned}$$

Recall the trick used in the case of linear momentum that

$$\sum_{\substack{ij \\ i \neq j}} \vec{r}_i \times \vec{f}_{ij} = \sum_{\substack{ij \\ i \neq j}} \vec{r}_j \times \vec{f}_{ji}$$

so that

$$\sum_{\substack{ij\\i\neq j}} \vec{r}_i \times \vec{f}_{ij} = \frac{1}{2}\sum_{\substack{ij\\i\neq j}} \left[\vec{r}_i \times \vec{f}_{ij} + \vec{r}_j \times \vec{f}_{ji}\right]$$

$$= \frac{1}{2}\sum_{\substack{ij\\i\neq j}} \left[\left(\vec{r}_i - \vec{r}_j\right)\times \vec{f}_{ij}\right] \quad \left(\vec{f}_{ji} = -\vec{f}_{ij} \text{ by Newton's III Law}\right)$$

$$= 0 \text{ if } \vec{f}_{ij} \parallel \left(\vec{r}_i - \vec{r}_j\right)$$

Under these conditions, i.e. *if the force between the particles is along the line joining them* , we get

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{f}_{i\,ext}$$
$$= \vec{\tau}_{ext}$$

Thus if $\vec{\tau}_{ext} = 0$ then $\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant}$. Thus is the law of *conservation of total angular momentum*. In the next lecture we will do a few example of its application.

We now conclude this lecture by listing the following points that we have learnt:

1. A rigid body needs six parameters to describe its general motion; three for translation and three for rotation,
2. Dynamics of rigid body is governed by its angular momentum,
3. The angular momentum satisfies the equation

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

under the condition that the internal forces satisfy Newton 's IIIrd law and an additional condition that $\vec{f}_{ij} \parallel \left(\vec{r}_i - \vec{r}_j\right)$

4. $\vec{L} = \vec{L}_{CM} + \vec{L}_{about\,CM}$

Lecture 19
Rotational dynamics II: Rotation about a fixed axis

We saw in the previous lecture on rigid bodies that a rigid body in general requires six parameters to describe its motion, and the dynamic of a rigid body is determined through its angular momentum $\vec{L}$ that satisfies the equation $\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$, where $\vec{\tau}_{ext}$ is the applied torque on the body. Further, $\vec{\tau}_{ext} = 0$ means that $\vec{L}$ is a constant.

In this lecture I start with an example of the conservation of angular momentum involving two particles. I again show that a direct application of Newton 's laws and a solution through the conservation of angular momentum give the same answer.

Example 1: There is a rigid massless rod of length *b* held at point *O* carrying a mass m_2 at its other end. Let the y-coordinate of m_2 be *a*. Another mass m_1 comes parallel to the x-axis and hits m_2 and the two masses get stuck together (see figure 1). Question is at what speed will the rod rotate?

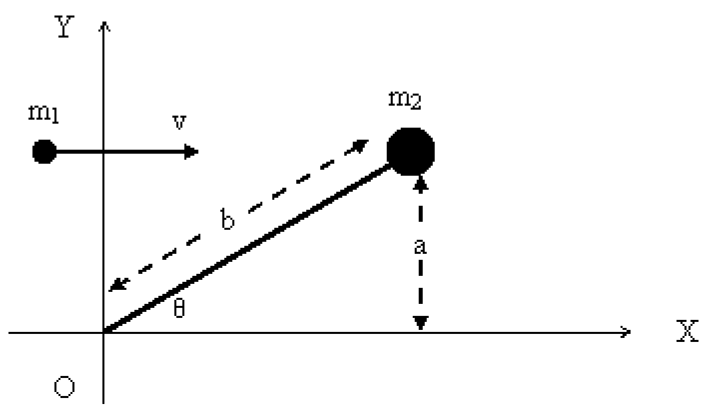


Figure 1

Let us apply the conservation of angular momentum to the system of two masses about point *O*. This is because the only external force acts at *O* so the torque about O is zero and therefore the angular momentum about *O* is conserved. Since the particles are moving in the *xy* plane, their angular momentum is going to be in the *z* direction. So we write the unit vector explicitly and work in terms of numbers (both positive and negative) only. Assume that the angular velocity of the rod after the mass m_1 gets stuck with it is ω. To apply angular momentum conservation we calculate the angular momentum of the system before and after collision and equate them.

Initial angular momentum about O $= m_1 va$

Final angular momentum $= (m_1 + m_2)b^2 \omega$

Equating the two gives

$$\omega = \frac{m_1 va}{(m_1 + m_2)b^2}$$

Let us now see if the conventional force analysis also gives the same answer. The incoming mass m_1 comes in with momentum m_1v. Now after m_2 is hit, it cannot have any movement parallel to the rod because the rod is rigid, i.e. the rod is capable of generating enough tension (impulse) in it to make the component of momentum parallel to the rod zero. On the other hand, there is no force perpendicular to the rod so the momentum component $p_\perp$ in that direction remains unchanged after the hit. Now

$$p_\perp = m_1 v \sin\theta = \frac{m_1 va}{b}$$

After the masses get stuck together, $p_\perp$ remains the same. Thus the new speed v' acquired by the masses will be such that

$$(m_1 + m_2)v' = \frac{m_1 va}{b} \quad \text{or} \quad v' = \frac{m_1 va}{(m_1 + m_2)b}$$

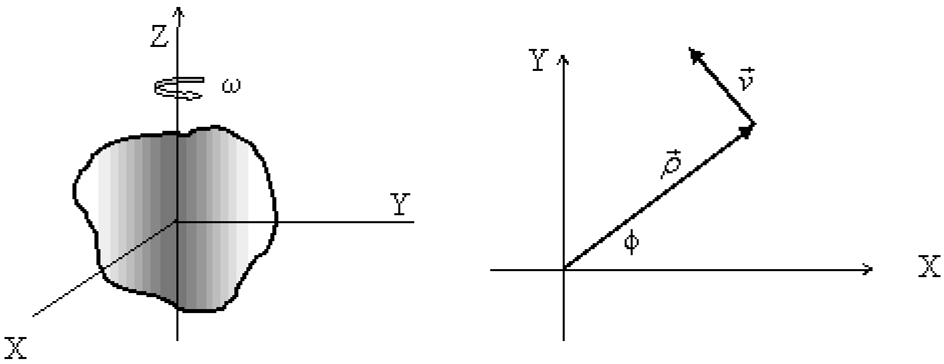
This gives

$$\omega = \frac{v'}{b} = \frac{m_1 va}{(m_1 + m_2)b^2}$$

which is the same as obtained by angular momentum conservation. Thus again showing the equivalence of the two methods.

With all this preparation, let us now start with the simplest motion of a rigid body that is the rotation of a rigid body about an axis fixed in space. So the axis is neither translating nor rotating. Without any loss of generality, let us call this axis the z-axis. In this case the body has only one degree of freedom and the only variable that we need to describe the motion of the body is the angle of rotation about the axis. Further, the only relevant component of angular momentum in this case is the component along the z-axis. Note that there may be other components of angular momentum but their change is accounted for by torques applied on the axis to keep it fixed in space. Calculation of such torques will be discussed in later lectures. Suffices here is to say that these torques arise out of the constraint forces that enforce the constraint of the axis being fixed in space.

Shown in figure 2 is a rigid body rotating about the z-axis with an angular speed ω. Also shown there is the position and velocity vector of one of its constituent particles of mass m_i in a plane perpendicular to the rotation axis. We wish to calculate the z component of the angular momentum.



A rigid body rotating about the z-axis (left) and the position and velocity vectors of a point in it (right) shown in the plane perpendicular to the z-axis

Figure 2

The z component will be given as

$$L_Z = \sum_i m_i (\vec{r}_i \times \vec{v}_i)_Z = \sum_i m_i \left(x_i v_{iy} - y_i v_{ix}\right)$$

For a particle at distance ρ_i from the z-axis and its radius vector making an angle Φ_i from the x-axis

$$x_i = \rho_i \cos\phi_i \qquad v_{ix} = -\varpi\rho_i \sin\phi$$
$$y_i = \rho_i \sin\phi_i \qquad v_{iy} = \varpi\rho_i \cos\phi$$

so that

$$L_Z = \left(\sum_i m_i \rho_i^2\right)\varpi$$

Calling $I_Z = \left(\sum_i m_i \rho_i^2\right)$ the moment of inertial about the rotation axis, we can write

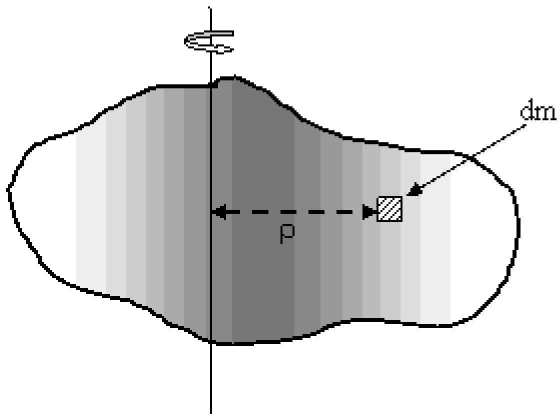
$$L_Z = I_Z \varpi$$

Depending on the direction of ω, angular momentum about an axis could have negative or positive values because it is a vector quantity. The convention we take is the right-hand convention; Let the thumb of one's right hand point in the positive z direction; if the rotation of the body is in the same (opposite) direction as the fingers, ω is positive (negative).

Having defined the moment of inertia about an axis, we make a few comments on it. First thing we notice about it is that it depends on the perpendicular distance of point masses from the axis of rotation. So no matter where we take the origin of the coordinate system, the moment of inertia of a rigid body about an axis is always going to be the same. Secondly, for continuously distribute mass moment of inertia is calculated as the integral

$$I_Z = \int \rho^2 dm$$

where ρ is the perpendicular distance of a small mass element dm taken in the body (see figure 3).



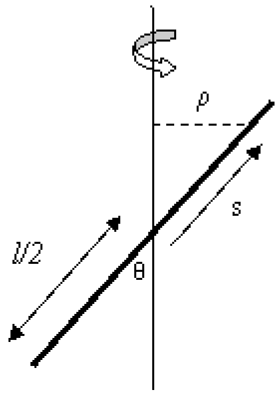
A small mass element dm in a rigid body at a distance ρ from the axis of rotation

Figure 3

Finally, for planar objects the moment of inertia is the same as the second moment of an area except that the area is replaced by the mass.

We now calculate moment of inertia of some objects.

A rod at an angle from the axis of rotation passing through its centre: This is shown in figure 4. The length of the rod is l and its mass m . It is at an angle θ from the axis of rotation.



A rod at an angle from the axis of rotation

Figure 4

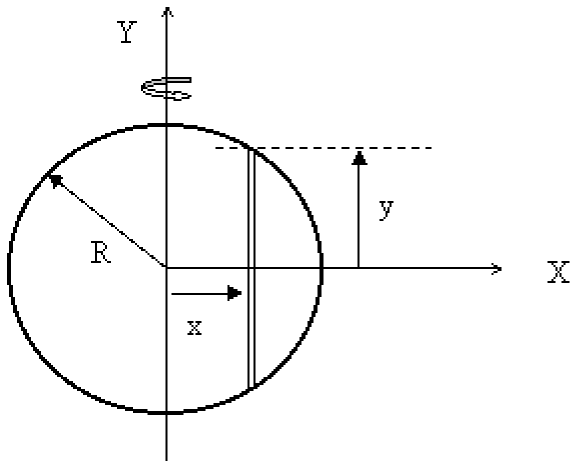
We take a small mass element of length *ds* at a distance *s* from the origin. It is at a distance $\rho = s\sin\theta$ from the axis of rotation. Then

$$\begin{aligned} I &= \int \rho^2 dm \\ &= \int_{-l/2}^{l/2} s^2 \sin^2\theta \frac{m}{l} ds \\ &= \frac{ml^2}{12}\sin^2\theta \end{aligned}$$

Thus for a rod rotating about its perpendicular passing through its centre is $\frac{ml^2}{12}$.

Exercise: Calculate the moment of inertia of a disc rotating about an axis passing through its centre and perpendicular to it.

Moment of inertia of disc about one of its diameters: Shown in figure 5 is a disc of mass *M* and radius *R* rotating about its diameter which lies on the y-axis.



A disc rotating about its diameter

Figure 5

To calculate the moment of inertia I take a strip of lengths width *dx* at distance x from the y-axis, the axis of rotation. Its mass is $dm = \frac{M}{\pi R^2} 2y\,dx$ (see figure 5). Thus

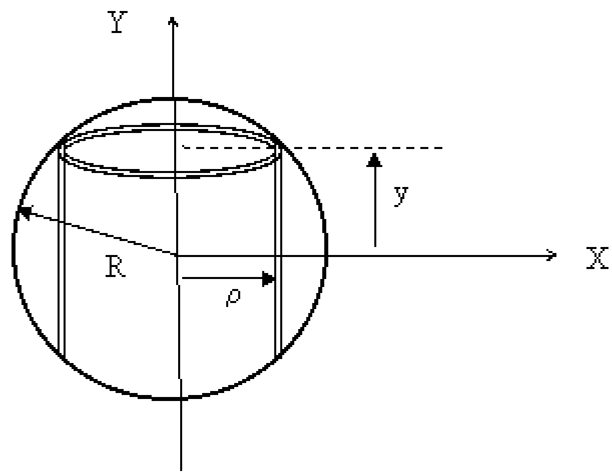
$$
\begin{aligned}
I &= \int x^2 dm \\
&= \int_{-R}^{R} x^2 \frac{M}{\pi R^2} 2y\,dx \\
&= \frac{2M}{\pi R^2} \int_{-R}^{R} x^2 \sqrt{R^2 - x^2}\,dx
\end{aligned}
$$

The integration can be carried out easily by substituting $x = R\cos\theta$ and gives

$$I = \frac{MR^2}{4}$$

Moment of inertia of a sphere about one of its diameters: A sphere of mass *M* and radius *R* is shown in figure 6. To calculate its moment of inertia, we take a cylindrical shell of radius ρ and thickness $d\rho$ (see figure 6). The mass of this shell is given by

$$dm = \frac{M}{4\pi R^3/3} \times 2\pi\rho d\rho \times 2y$$
$$= \frac{3M}{R^3}\sqrt{R^2-\rho^2}\,\rho d\rho$$



A sphere rotating about its diameter

Figure 6

Therefore the moment of inertia is

$$I = \int \rho^2 dm = \frac{3M}{R^3}\int_0^R \rho^3\sqrt{R^2-\rho^2}\,d\rho$$

By substituting $\rho = R\cos\theta$, this is an easy integral to perform and gives the result

$$I = \frac{2}{5}MR^2$$

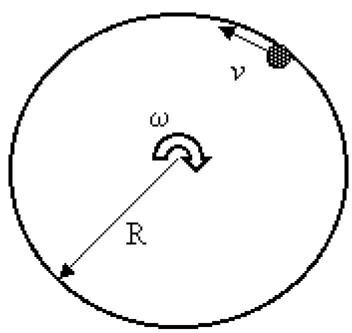
Let us now recapture what we have done so far. We have looked at the angular momentum of a body rotating about a fixed axis. We find that angular momentum L_Z about an axis (denoted as the z-axis) is given as $L_Z = I_Z\,\omega$ and, depending upon the sense of rotation, can take positive as well as negative values. We have also calculated I_Z for some standard objects about an axis. We now go on to study the equation of motion satisfied by L_Z . The equation satisfied by $\boldsymbol{L_Z}$ is

$$\frac{dL_Z}{dt} = \tau_Z$$

where τ_z is the component of the *external* torque along the axis of rotation. If the external torque is zero, the angular momentum is conserved. You can observe the effect of conservation of angular momentum easily at home.

Sit on a revolving chair holding a brick (or something similar) in each of your hands and keep your arms stretched. Start revolving the chair and then pull your arms in. You will observe that you start revolving much faster. This happens because when you pull the arms in, the masses that you are holding come closer to the axis of rotation resulting in a reduction in the value of the moment of inertia. However, since there is no external torque on the system, the angular momentum cannot change. Thus if the moment of inertia decreases, the angular speed must increase in order to keep $L = I\omega$ constant. This is precisely what you observe. You should also repeat the experiment holding different weights. When do you observe the rotational speed to increase the largest? Let us now solve an example of applying the angular momentum conservation principle.

Example: A man starts walking on the edge of a circular platform with a speed v with respect to the platform (see figure 7). The platform is free to rotate. What is the rotational speed of the platform? Mass of the platform is M , its radius is R and the mass of the man is m .



A man of mass m walks on the edge of a circular platform of mass M and radius R. the platform in turn starts rotating.

Figure 7

Since there is no external torque, the angular momentum of the system about the axis of rotation must be conserved. Thus as the man starts walking, the platform starts rotating the other way. Since the speed of man with respect to the platform is v , his speed in the ground frame would be $(v - \omega R)$. Thus the angular momentum of the man is

$$mR(v - \omega R)$$

At the same time, the angular momentum of the platform is

$$-\frac{1}{2}MR^2\omega$$

where the minus sign shows that the angular momentum of the platform is in the direction opposite to that of the man's angular momentum. By conservation of angular momentum

$$mR(v-\omega R)-\frac{1}{2}MR^2\omega=0$$

which gives

$$\omega=\frac{v/R}{\left(1+\frac{M}{2m}\right)}$$

Having learnt about the angular momentum, its equation of motion and the conservation of angular momentum for rotations about a fixed axis, we now discuss the kinetic energy and the work-energy theorem for a rigid body rotating with angular speed w about a fixed axis.

Kinetic energy and work-energy theorem for a rigid-body rotating about a fixed axis: The kinetic energy of a rigid body rotating with angular speed ω is obtained by calculating the energy of small mass element in the body and adding it up. This mass element is rotating in a plane perpendicular to the axis of rotation. This gives (using the notation of figure 2)

$$\begin{aligned} K.E. &= \frac{1}{2}\sum_i m_i v_i^2 \\ &= \frac{1}{2}\left(\sum_i m_i \rho_i^2\right)\omega^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned}$$

The corresponding work-energy theorem for the motion considered here is that the change in kinetic energy is equal to the work done on the body. Let us first calculate the work done on a body, which can only rotate about an axis, when an external force is applied on it. To do this, I would first like you to prove a result (look at figure 2 for reference): when a body rotates by an angle $\Delta\theta$ about an axis in the unit vector direction $\hat{n}$, the corresponding change in position of a particle in the body at position vector $\vec{r}_i$ is

$$\Delta\vec{r}_i = (\hat{n}\times\vec{r}_i)\Delta\theta$$

The total work done on the body by a net external force composed of forces $\vec{f}_i$ acting at each point is

$$\Delta W = \sum_i \vec{f}_i \cdot \left(\hat{n} \times \vec{r}_i\right) \Delta\theta$$

By using $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{A} \times \vec{C})$, we can write the work done as

$$\begin{aligned} \Delta W &= \sum_i \hat{n} \cdot \left(\vec{r}_i \times \vec{f}_i\right) \Delta\theta \\ &= \hat{n} \cdot \left(\sum_i \left(\vec{r}_i \times \vec{f}_i\right)\right) \Delta\theta \\ &= \hat{n} \cdot \vec{\tau} \Delta\theta \\ &= \tau_Z \Delta\theta \end{aligned}$$

where τ_Z is the component of the external torque along the axis of rotation. Thus the total work done is

$$W = \int \tau_Z d\theta$$

Now the work energy theorem can be expressed as follows:

$$\Delta K.E. = \int \tau_Z d\theta$$

This pretty much concludes what all I have to say about the rotations about a fixed axis. One question that may be asked at this point is: Why is it what describing dynamics in term of angular momentum, torque etcetera rather than momentum and force is more useful in discussing rotational motion. This is because in rotational motion, force, momenta etcetera are distributed and taking their moments by considering the angular momenta and torques automatically takes care of this distribution. We conclude this lecture by drawing a comparison between linear and rotational motion about a fixed axis.

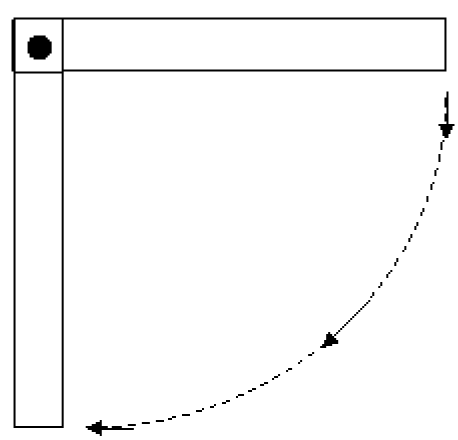
Linear motion	**Rotational motion about a fixed axis**
Momentum p	Angular momentum L
$\frac{dp}{dt} = F$	$\frac{dL}{dt} = \tau$
Impulse $\int F dt = \Delta p$	Impulse $\int \tau dt = \Delta L$
$K.E. = \frac{1}{2} m v^2$	$K.E. = \frac{1}{2} I \omega^2$
$\Delta K.E. = \int F dx$	$\Delta K.E. = \int \tau d\theta$

This correspondence will help in understanding and getting relationships to solve most of the problems involving rotations about a fixed axis, particularly if you have solved many problems involving linear momentum.

Rigid body dynamics III: Rotation and Translation

We have seen in the past two lectures how do we go about solving the rigid body dynamics problem by considering the rate of change of angular momentum. In the previous lecture, we concentrated on rotation about a fixed axis and solved problems involving conservation of angular momentum about that axis. In this lecture we consider what happens where an external torque is applied and also when the axis is allowed to translate parallel to itself.

Let us first take the case when the axis is stationary and a torque is applied. Take for instance your pen or a scale and hold it lightly at one of its ends so as to pivot it there. Raise the other end so that the scale is horizontal and then leave it. You will see that the scale swings down. I would like to calculate the speed of its CM when the scale is vertical after being released from horizontal position (see figure 1). Assume that there is no loss due to friction. In this case I will solve this problem in two ways and also comment on a wrong way.



A scale pivoted at one end and released from horizontal position swings down

Figure 1

I take the mass of the scale to be *m* and its length *l*. Then its moment of inertia about one of its ends is $I = \dfrac{ml^2}{3}$.

I first solve the problem using energy conservation. Since there is no loss due to friction the total mechanical energy is conserved. Therefore the total mechanical energy is conserved. Let us take the potential energy to be zero when the scale is horizontal. Since the scale starts with zero initial angular speed, its total mechanical energy is zero. When the scale reaches the vertical position, its CM has moved down by a distance $l/2$ so its potential energy is $-mgl/2$. If its angular speed at that position is ω, then by conservation of energy

$$\frac{1}{2}I\omega^2 - \frac{mgl}{2} = 0$$

which gives

$$\omega = \sqrt{\frac{3g}{l}}$$

I now solve the problem by a direct application of torque equation. When the scale makes an angle θ from the horizontal (see figure 2), the torque on it is given as

$$\tau = \frac{mgl}{2}\cos\theta$$

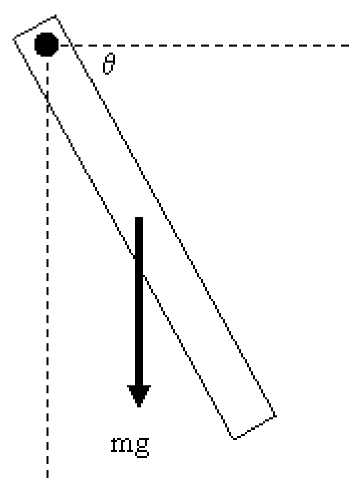


Figure 2

The angular momentum-torque equation then gives

$$I\frac{d\omega}{dt} = \frac{mgl}{2}\cos\theta$$

Substituting $\omega = \frac{d\theta}{dt}$ and the value of *I* from above this leads to

$$\frac{d^2\theta}{dt^2} = \frac{3g}{2l}\cos\theta$$

This equation cannot be integrated with respect to time directly. Recall from the proof of work-energy theorem that in such situations we change transform the equation to write it in terms of the displacement variable, which is the angle in this case. So we write

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{d\theta}{dt}\right) = \frac{1}{2}\frac{d}{d\theta}\left(\dot{\theta}^2\right)$$

to write the equation above as

$$\frac{1}{2}\frac{d\dot{\theta}^2}{d\theta} = \frac{3g}{2l}\cos\theta$$

Integrating this equation then gives

$$\dot{\theta}^2 = \frac{3g}{l}\sin\theta$$

For $\theta = \pi/2$ this gives the same answer as obtained earlier. If you have noticed, what we have done here is actually used the work-energy theorem

You may ask at this point: wouldn't the correct way of solving this problem be to equate the kinetic energy of the CM to the change in the potential energy. This would lead to

$$\frac{1}{2}mv_{CM}^2 = \frac{mgl}{2}$$

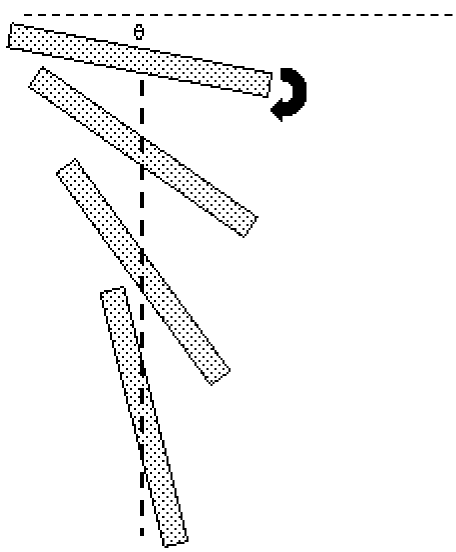
$$v_{CM} = \sqrt{gl} \quad \text{and} \quad \omega = \frac{v_{CM}}{l/2} = 2\sqrt{\frac{g}{l}}$$

The reason why this answer is incorrect is the following. Recall from our previous lecture that the most general motion of a rigid body is a translation plus a rotation. So while it is true that the CM is moving, the scale is also rotating at the same time. We represent the combination of the two motions as a translation of the CM and a rotation about an axis passing through the CM. Why we split the motion of the scale as a combination of the translation of its CM and a rotation about the CM - and not that of any other point in the body - will be discussed in detail below. For now it is sufficient to say that by doing so the kinetic energy can be written conveniently as (KE of the CM plus KE about the CM). So the true K.E of the scale is

$$\frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

where $I_{CM} = \frac{ml^2}{12}$ is the moment of inertia about the CM. Using the relationship $v_{CM} = \frac{l}{2}\omega$ this gives the same kinetic energy as that used above in applying the energy conservation method. This correct approach then gives the same answer as obtained above.

An interesting problem related to the one solved above is as follows. Sometimes if a book you are holding slips out your hand, it usually falls with its upper face down (see figure 3). You can try this at home and see for yourself. In fact there is an interesting book which has a title based on this observation. It is entitled "Why toast lands jelly side down" and is authored by Robert Ehrlich (Universities press, Hyderabad 1999). Let us try to understand this observation.



A book falling down after slipping out of one's hand. Notice that the centre of the book is shown to be coming down in a straight line while the book rotates at a constant angular speed..

Figure 3

When the book falls its angular acceleration α immediately after it slips off the hand is calculated approximately as given below

$$I\frac{d\omega}{dt} = \frac{mgl}{2} \quad \Rightarrow \quad \frac{ml^2}{3}\frac{d\omega}{dt} = \frac{mgl}{2}$$

$$\Rightarrow \quad \alpha = \frac{d\omega}{dt} = \left(\frac{3g}{2l}\right)$$

Here *m* is the mass of the book and *l* its length. I call it an approximate expression because in our calculation we have assumed the book to be in horizontal position. It will slip off when $\tan\theta = \mu$ or for small angles $\theta \sim \mu$, where *m* is the coefficient of friction between the book and the hand. Starting with zero initial angular speed, let the angular speed of the book when it slips out of the hand be ω . Then

$$\omega = \sqrt{2\alpha\theta} = \sqrt{\frac{3\mu g}{l}}$$

Taking $\mu = 0.5$, $g = 10 ms^{-2}$ and $l = 20cm = 0.2m$, we get

$\omega = 8.7$ rad s^{-1}

After the book has come out of the hands, there is no external torque on it about its CM so it falls rotating with a constant angular speed of about 8.7rad s^{-1}. Keep in mind that the sense and amount of rotation of a rigid body is the same irrespective of the point about which its rotation is considered. So although before slipping out of the hand, I did the calculation for its angular speed taking its edge on the hand as the axis, after it comes out of the hand, I consider its motion as the translation of its CM and rotation about its CM. Let us stake a typical height of about 1m from which the book falls. Then the time it takes to reach the ground is

$$t = \sqrt{\frac{2h}{g}} = 0.45s$$

Thus the angle through which the book rotates by the time it reaches the ground is

$$\theta = \omega t = 8.7 \times 0.45 = 3.9rad \approx 225°$$

If we add to this angle the initial rotation of $\theta = \mu = 0.5$, the angle increases to about 250°. The angle of rotation of course varies in a range but it is around 180°.You see that the book has just the right angular speed and the time of fall for it to turn by around 180°. That is precisely what we observe.

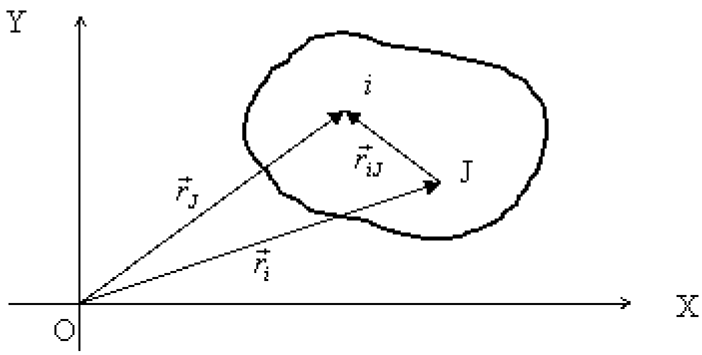
Rotation of a rigid body combined with translation of the axis parallel to itself: Let us now introduce translation of the rotational axis parallel to itself - it may even accelerate - and ask what kind on motion is going to follow. So for example there may be a rod on a horizontal table and is hit by an impulse one end, and we may be interested in its subsequent motion. I general it could be a rigid body of general shape on which we apply a force. We split the motion into a translation of the CM of the body and rotation about an axis passing through the CM. By doing so the equation of motion for the translational motion of the CM is very easy. It is

$$\frac{d\vec{p}_{CM}}{dt} = M\vec{a}_{CM} = \vec{F}_{applied}$$

Here $\vec{p}_{CM}$ is the total momentum of the body; *M* is its mass; $\vec{a}_{CM}$ the acceleration of the *CM* and $\vec{F}_{applied}$ the total applied force. With this equation we know how the CM of the body translates. Next we wish to find the rotation of the body with respect to an axis passing through the CM (recall that the most general motion of a rigid body is translation of a point and rotation about that point). But the question is: can we apply

$$\frac{dL_{CM}}{dt} = \tau_{applied,CM}$$

where L_{CM} is the angular momentum about the CM and $\tau_{applied,CM}$ is the applied torque about the axis of rotation passing through the CM. I raise this question because in general the CM will also be accelerating and therefore with respect to the CM, there will be a fictitious force that may also give rise to a fictitious torque which is in addition to the applied torque $\tau_{applied,CM}$. However, it is easy to see that such a fictitious torque about the CM will always be zero. This is because the fictitious force effectively acts at the CM itself. Because of this reason, there is one more point about which the torque due to the fictitious force vanishes: this is the point that accelerates towards the CM. Thus the equation above can be applied safely about these two points. There is also a third point about which the above equation is valid. This is the point that does not accelerate at all. Let me now prove these statements.



A rigid body that is both translating and rotating. Two points in the body are shown by J and i.

Figure 4

Shown in figure 4 is a rigid body performing a general motion, i.e. it is both translating as well as rotating. For convenience we have shown the body in two dimensions. Two points *J* and *i* of the body are also shown. These points are also moving with the body. We now calculate the rate of change of the angular momentum about point *J* . This is done below.

$$\vec{L}_J = \sum m_i \vec{r}_{iJ} \times \vec{v}_{iJ}$$

$$\frac{d\vec{L}_J}{dt} = \sum m_i \vec{r}_{iJ} \times \frac{d\vec{v}_{iJ}}{dt}$$

$$= \sum m_i \vec{r}_{iJ} \times \vec{a}_{iJ}$$

where all the terms have their standard meaning and the subscript (*iJ*) denotes the quantity being calculated for point *i* with respect to point *J* . Denoting the velocity and acceleration of point *i* about the origin *O* as $\vec{v}_i$ and $\vec{a}_i$, respectively, and that of *J* as $\vec{v}_J$ and $\vec{a}_J$, we have

$$\vec{a}_{iJ} = \frac{d}{dt}(\vec{v}_i - \vec{v}_J) = \vec{a}_i - \vec{a}_J \quad \text{and} \quad \frac{d\vec{L}_J}{dt} = \sum_i m_i (\vec{r}_{iJ} \times \vec{a}_i) - \left(\sum_i m_i \vec{r}_{iJ}\right) \times \vec{a}_J$$

With $m_i \vec{a}_i = \vec{f}_{i,applied}$ and $\sum m_i \vec{r}_{iJ} = M\vec{R}_{CM,J}$, where $\vec{R}_{CM,J}$ is the position vector of the CM with respect to *J* , we get

$$\frac{d\vec{L}_J}{dt} = \sum_i \left(\vec{r}_{iJ} \times \vec{f}_{i,applied}\right) - M\vec{R}_{CM,J} \times \vec{a}_J$$

$$= \vec{\tau}_{applied} - M\vec{R}_{CM,J} \times \vec{a}_J$$

If we want the rate of change of the angular momentum to depend only on the applied torque calculated about J, we should have

$$M\vec{R}_{CM,J} \times \vec{a}_J = 0$$

That will happen under the following three conditions:

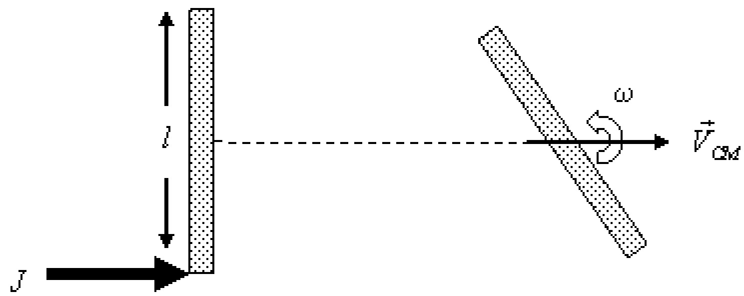
$$(I)\quad \vec{a}_J = 0, \text{ i.e. J is moving uniformly,}$$
$$(II)\quad \vec{a}_J \parallel \vec{R}_{CM,J}, \text{ i.e. J is accelerating towards the CM,}$$
$$(III)\quad \vec{R}_{CM,J} = 0, \text{ i.e. point J is the CM}$$

I have just shown you that irrespective of the whether point *J* is accelerating, rotating or performing some general motion, the equation

$$\frac{d\vec{L}_J}{dt} = \vec{\tau}_{applied,J}$$

can be applied about *J* if it satisfies one of the three conditions obtained above. Notice that in under these conditions the right-hand side has only the externally applied torque. Thus if we choose one of these points to apply the angular momentum-torque equation, we do not have to worry about any fictitious torques arising because we are sitting on an accelerating point. We have been applying the angular momentum-torque equation about points satisfying condition *I* above; it includes stationary points also. Of the other two points, it is always safer to apply the equation about the CM (condition *II*). This is because of the difficulty in ensuring that a point is accelerating towards the CM (condition *III*), although in some situations it may be easy. We will discuss one such case below. We now solve some simple examples to illustrate what we have learnt above.

Example 1: A uniform rod of mass *m* and length *l* is on a smooth horizontal table (friction = 0) and is hit at one of its ends so that an impulse *J* is imparted to it in its perpendicular direction (see figure 5). What is its subsequent motion?



A rod being hit at one of its ends giving it an impulse J(left) and its position and orientation of the rod some time after the impact(right).

Figure 5

As the rod is hit, its CM will start moving with a velocity

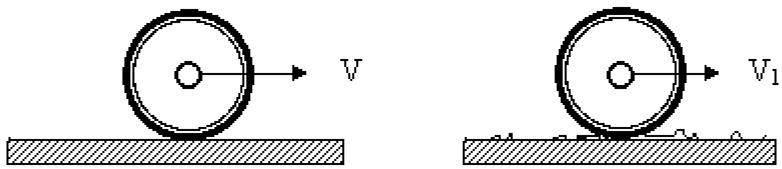
$$V_{CM} = \frac{J}{m}$$

At the same time the rod also starts rotating. Although the CM will be accelerating during the impact, we can apply the angular momentum-torque equation about it with only external torque in the equation. If the angular speed of the rod after the impact is ω, it is given by

$$\frac{l}{2}J = I_{CM}\omega \quad \Rightarrow \quad \omega = \frac{lJ}{2\left(ml^2/12\right)} = \left(\frac{6J}{ml}\right)$$

Note that in the sentence above, I have said 'angular speed of the rod' and not 'angular speed of the rod about the CM because the sense and amount of rotation about any point in the body is the same, as was discussed in a previous lecture. The position and orientation of the rod some time after the impact is also shown in figure 5.

Example 2: A wheel of mass m and radius R is sliding on a smooth surface (No rolling) with speed V. It then hits a very rough surface so that it starts rolling (see figure 6). What is it rolling speed?



A wheel sliding on a smooth surface (left) hits a rough surface (right) and starts rolling.

Figure 6

Let the rolling speed of the wheel be V_1 . As soon as the wheel hits the rough surface, it gets an impulse J at its point on the surface in the direction opposite to its velocity. This reduces its speed and also makes it rotate. It rolls if the speed V_1 of its CM is equal to ωR , where ω is the rolling speed it gains after hitting the rough surface. The change in the CM speed is given by

$$V - V_1 = \frac{J}{M}$$

Applying the angular-momentum torque equation about the CM, we get

$$mR^2\omega = JR \quad \Rightarrow \quad J = mR\omega$$

With the condition of rolling, $V_1 = \omega R$, the above two equations give

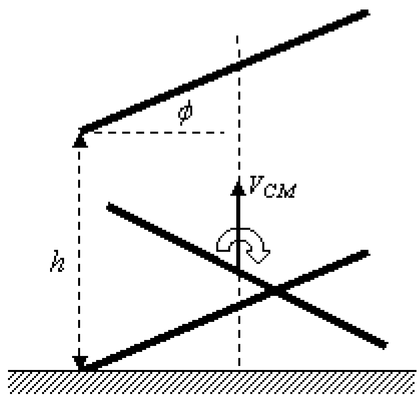
$$V_1 = \frac{V}{2}$$

I would like you to repeat the same exercise for a disc.

The problem can also be solved by applying conservation of the point of impact on ground, because the impulse gives zero torque about that point. The initial angular momentum of the wheel with respect to that point is mVR . The final angular momentum is (angular momentum of the CM plus angular momentum about the CM). This comes out to be $(mV_1R + mR^2\omega)$. Equating this to mVR and using the rolling condition gives the same answer as above. A warning: keep in mind that the torque is being taken with respect to the point on ground and not the point on the wheel that is touching the ground. Doing that will not be correct because at the time of impact the point on the wheel is accelerating in the direction opposite to $\vec{V}$.

I now solve a problem that involves, in addition to the equations above, energy conservation also.

Example 3: A rod of mass *m* and length *l* is held making an angle Φ from the horizontal at a height *h* from the floor (see figure 7). When dropped from rest, what will be its linear and angular speed after it rebounds from the floor? Assume no energy is lost during the impact with the floor.



A rod dropped from a height rebounds from the floor and starts rotating. The CM moves only in vertical direction

Figure 7

When the rod hits the floor, it receives an impulse *J* from the ground in the vertically up direction. Although the rod is also being acted upon by its weight, we neglect its effect during impact (see discussion in the lecture on momentum). Since all the forces are in the vertical direction, the CM of the rod also moves only vertically. Before hitting the floor, the speed of the CM is $\sqrt{2gh}$ and the angular speed of the rod is zero. Let the rebound speed of the CM be *V* and the angular speed of the rod after rebounding be ω. Then similar to the example above, these quantities are related as (keep in mind that we are dealing with vector quantities so their signs have to be properly accounted for)

$$mV + m\sqrt{2gh} = J$$

$$\frac{ml^2}{12}\omega = J\frac{l}{2}\cos\phi$$

These are two equations for three unknowns: V, ω and J. We therefore need one more equation. This is provided by energy conservation. We express the kinetic energy of the rod after it rebounds as the sum of the kinetic energy of its CM and the kinetic energy about its CM. Thus immediately after the impact, energy conservation gives

$$mgh = \frac{1}{2}mV^2 + \frac{1}{2}\frac{ml^2}{12}\omega^2$$

Now we have three equations that can be solved for the three unknowns. This is left for you to do.

Question you might now ask is if we could use the principle angular momentum and energy conservation directly to solve this problem in a manner similar to what we did at the end of the last example. We would like to apply the conservation of angular momentum about the point of impact on the ground because torque due to the impulse about this point vanishes. Although there is another external force - the weight of the rod - acting on the system, its effect during the impact can be ignored because very short duration of impact. Thus we can say that the angular momentum about the point of impact is conserved. This gives (left as an exercise for you)

$$m\sqrt{2gh}\,\frac{l}{2}\cos\phi = \frac{ml^2}{12}\omega - mV\frac{l}{2}\cos\phi$$

or

$$\left(m\sqrt{2gh} + mV\right)\frac{l}{2}\cos\phi = \frac{ml^2}{12}\omega$$

This is the same equation that is obtained by combining the first two equations above. Thus we obtain the same answer by this method also.

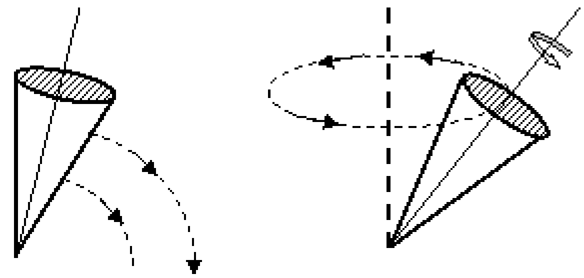
I end this lecture by giving you an exercise.

Exercise: A disc of mass m and radius R is made to roll on a rough surface by applying a force F at its centre. If it does not slip on the floor, i.e. it does pure rolling, find its acceleration by applying methods developed in this lecture.

Rotational dynamics IV: Angular velocity and angular momentum

In the previous three lectures, we have dealt primarily with rotation about a fixed axis or an axis moving parallel to itself. What we saw in those lectures was that dynamics of a rigid body is described by $\left(\frac{d\vec{L}}{dt}\right) = \vec{\tau}_{external}$ and in the absence of $\vec{\tau}_{external}$ the angular momentum $\vec{L}$ is a conserved. In the case of fixed axis rotation, the relationship between the angular momentum and the angular speed was quite straight forward in that $L = I\omega$ and all that was done in those problems was to change the magnitude of ω to change L. But the rotational motion is much more interesting than that. For example $\vec{L}$ is a vector so it could change direction because of applied torque with or without its magnitude being affected. How the changing direction of $\vec{L}$ affects the orientation of a rigid body is one question we should answer if we wish to understand the motion of a rigid body. To start with, I want to point out to you that rotational motion is sometimes not what one would expect naively.

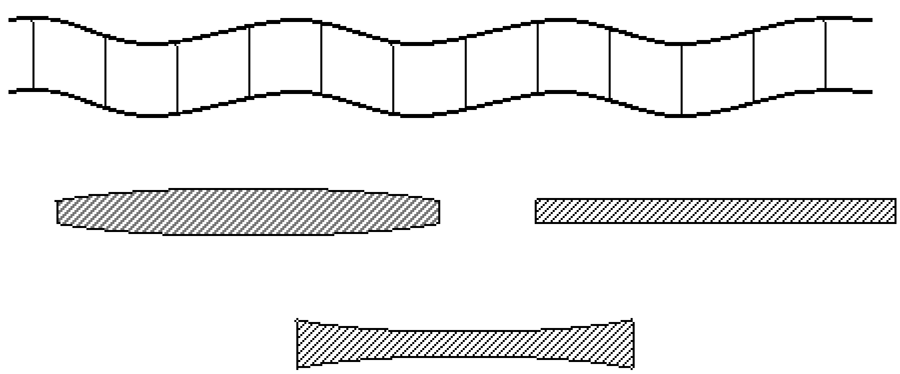
You must have played with a top. If it is not spinning and we try to make it stand on its pivot, it falls sideways. On the other hand, if it is given a spin and then put on its pivot point, it does not fall but starts to move about, what is called precession, a vertical axis passing through its pivot point. This is shown in figure 1. Obviously the precession of the top has something to do with its spin.



A top that is not spinning falls to the side (left) whereas a spinning top starts precessing (right).

Figure 1

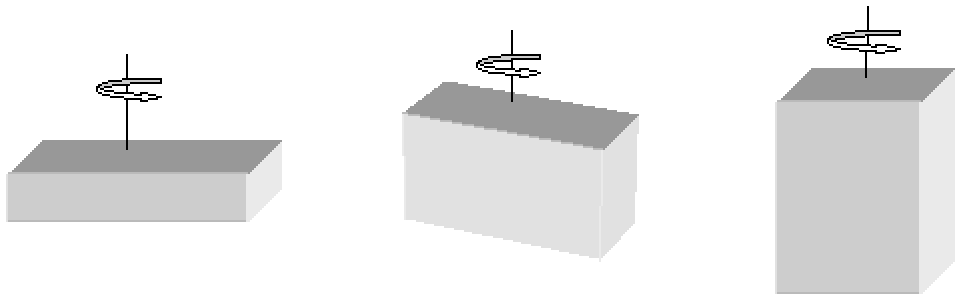
My second observation is from something that is seen in science museums. You can also make it easily in your local workshop. Take a track with many soft curves on it and let three different shape rollers roll on it. You may want to keep the track slightly tilted so that the rollers roll by themselves. Question is which of the rollers will be able to negotiate all the curves.



A curved track (top) as seen from above. Three rollers of different shapes are made to roll on it. Question is which one of these will negotiate all curves?

Figure 2

I make the third observation on a rectangular box of sweets (empty of course) or any similar box. Put a rubber-band around it so that its lid does not come off. Hold the box at a height with one of its faces perpendicular to the vertical, give it a spin and let it drop (see figure 3). Observe how its spin changes when it is falling down. You will find that in two out of three possible ways of holding the box, its spin will remain essentially unchanged whereas in one case it will start wobbling. On the other hand, if the box is dropped without giving it a spin, it comes down in the same orientation. What does the spin do to it? We wish to understand this.

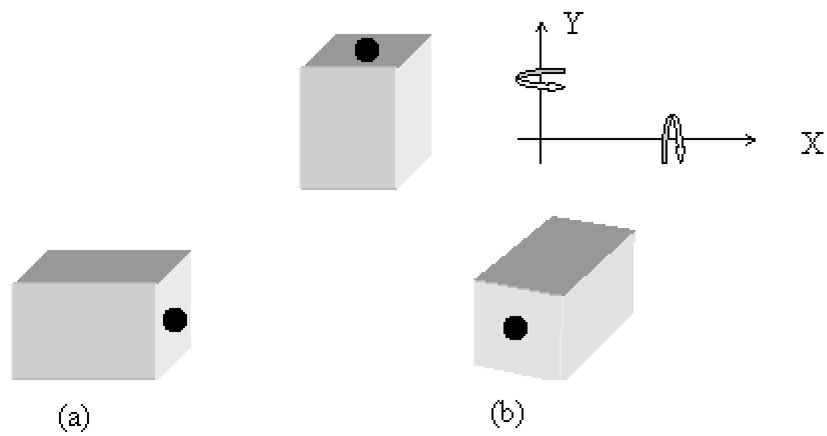


A box being given spin about different axes and being dropped from a height. In one case it wobbles while coming down.

Figure 3

In all three cases we see that when an object is given a spin its motion is very different compared to when it is not spinning. This happens because the angular momentum of the object due to its spin changes direction during the motion and the orientation of the body changes accordingly. So we now really have to get into the vector nature of angular momentum and relate it to the parameters - the angle and the angular speed / velocity - of the body. I develop this structure of three-dimensional rigid-body dynamics step-by-step. The first question we address in this development is if the angle of rotation θ can be expressed as a vector $\vec{\theta}$? And if the answer is yes, what is its direction?

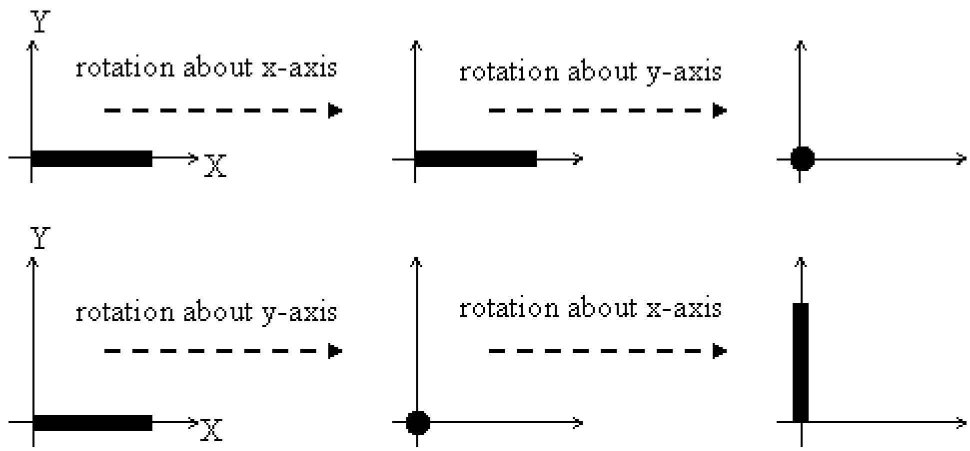
The answer to the question whether an angle of rotation can be treated as a vector is in the negative. This is because it fails to satisfy a fundamental property - that the addition of vectors is commutative - of vector addition. Thus if we make two rotations of angles θ_1 and θ_2 about two different axes, the end results will not be the same if the order of rotations is changed. This is depicted in figure 4 where I show a rectangular box that is to be rotated by 90° about the x and the y axes. The x and y axes are in the plane of the paper and pass through the centre of the box; the z-axis is coming out of the paper. The results are different if (a) I do the rotation about the x-axis first and then follow it with a rotation about the y-axis, and (b) I do the rotation about the y-axis first and then follow it with a rotation about the x-axis. Thus θ_1 and θ_2 cannot be treated as vectors because $\vec{\theta}_1 + \vec{\theta}_2 \neq \vec{\theta}_2 + \vec{\theta}_1$.



Results of two rotations – one about the x-axis by 90° counterclockwise and the second about the y-axis by 90° counterclockwise – performed on a box: (a) x-axis rotation that is followed by y-axis rotation, (b) y-axis rotation that is followed by an x-axis rotation.

Figure 4

Mathematically let us take a rod of length *l* lying along the x-axis with one of its ends at the origin so that the *(xyz)* coordinates of its other end are (*l*, *0*, *0*). Keeping its end at the origin fixed, the rod is rotated about the *x* and the *y* axes in the same manner as the box in figure 4. If rotated about the x-axis first the end still has coordinates (*l*, *0*, *0*). Now the rotation about the y-axis makes the rod align with the z-axis with the new coordinates of its end being *(0, 0, - l)* . Let us perform the rotations in the other order now. The first rotation is performed about the y-axis and makes the rod align with the z-axis with the new coordinates of its end being rod *(0, 0, - l)* . Now the rotation about the x-axis makes the rod align with the y-axis and the final coordinates of its end are *(0, l, 0)* . Thus we see that two rotations have absolutely different effect on the orientation of a body depending on their order. This is demonstrated in figure 5. The conclusion therefore is that *rotations in general cannot be treated as vectors* .



Different order of rotation about x and y axes leaves the rod being rotated with two different final orientations. In the upper set of rotations, first rotation of 90° is performed about the x-axis and is followed by a rotation of 90° about the y-axis. In the lower set, the order is interchanged.

Figure 5

Although rotations by a finite angle are no vector quantities, rotations by infinitesimal angles $\Delta\theta$ are. This also makes the derivative $\omega = \dfrac{\Delta\theta}{\Delta t}$ a vector quantity. We therefore call this quantity angular velocity rather than angular speed. Let me first show you through a simple example that infinitesimal rotations do satisfy the commutative property of vector addition and then go on to assign a direction to such rotations

Let me again take a rod lying along the x-axis with one end fixed at the origin and the other at *(l,0,0)*. However, this time I consider infinitesimal rotations about the *y* and the *z* axes. I do so because I want both the rotations to cause change in the orientation of the rod; first rotation about the x-axis does not do that. Before I present the calculations, I would like you to recall from the first lecture how different components of a vector change when the frame is rotated. I would be making use of those relationships now with one change: rotating a vector by an angle $\Delta\theta$ about an axis is same as viewing it from a frame rotated by the angle $-\Delta\theta$ about the same axis. I perform a rotation of the rod about the y-axis by an angle $\Delta\theta_y$ and that about the z-axis by angle $\Delta\theta_z$. Let me first consider the case of rotation about the y-axis that is followed by a rotation about the z-axis. Rotation of the rod about the y-axis gives the new coordinates of it free end as

$$x' = -0 \times \sin(-\Delta\theta_y) + l\cos(-\Delta\theta_y) \approx l$$

$$y' = 0$$

$$z' = 0 \times \cos(-\Delta\theta_y) + l\sin(-\Delta\theta_y) \approx -l\Delta\theta_y$$

Now rotate the rod about z-axis to get coordinates of its free end as

$$\boxed{\begin{aligned}&x'' \approx l\cos(-\Delta\theta_z) + 0\times\sin(-\Delta\theta_z) \approx l\\&y'' = -l\sin(-\Delta\theta_z) + 0\times\cos(-\Delta\theta_z) \approx l\Delta\theta_z\\&z'' = -l\Delta\theta_y\end{aligned}}$$

Let us now do it the other way. Rotation about the z-axis gives

$$\begin{aligned}&x' \approx l\cos(-\Delta\theta_z) + 0\times\sin(-\Delta\theta_z) \approx l\\&y' = -l\sin(-\Delta\theta_z) + 0\times\cos(-\Delta\theta_z) \approx l\Delta\theta_z\\&z' = 0\end{aligned}$$

Now give a rotation about the y-axis to get

$$\boxed{\begin{aligned}&x'' \approx -0\times\sin(-\Delta\theta_y) + l\cos(-\Delta\theta_y) \approx l\\&y' = l\Delta\theta_z\\&z'' = 0\times\cos(-\Delta\theta_y) + l\sin(-\Delta\theta_y) \approx -l\Delta\theta_y\end{aligned}}$$

When we compare the two boxed results above, we find that the coordinates of the end point of the rod come out to be the same. We conclude that two infinitesimal rotations will give the same final result irrespective of the order in which they are applied. Thus *infinitesimal rotations can be treated as vectors* . But what about the direction of rotation? To assign a direction, notice that the change in the position vector $\vec{r} = l\hat{x}$ of the end coordinate of the rod considered above can be written as

$$\begin{aligned}\Delta\vec{r} &= (\Delta\theta_y\hat{j} + \Delta\theta_z\hat{k})\times\vec{r}\\&= (\Delta\theta_z\hat{k} + \Delta\theta_y\hat{j})\times\vec{r}\\&= l\Delta\theta_z\hat{i} - l\Delta\theta_y\hat{k}\end{aligned}$$

where I have written the second line above to emphasize that the order in which infinitesimal rotations are performed does not affect the end result of these operations. The equations above suggest that an infinitesimal rotation about an axis be assigned a direction parallel to the axis following the right hand convention: If the thumb of the right hand points in the direction of the infinitesimal rotation, the movement of fingers gives the sense of rotation. With this definition, the change in the position vector of a point after it is rotated by an infinitesimal angle $\Delta\theta$ about an axis in the direction of unit vector $\hat{n}$ (sense of rotation given by right hand convention) is given as

$$\Delta\vec{r} = \Delta\theta\,\hat{n}\times\vec{r} = \Delta\vec{\theta}\times\vec{r}$$

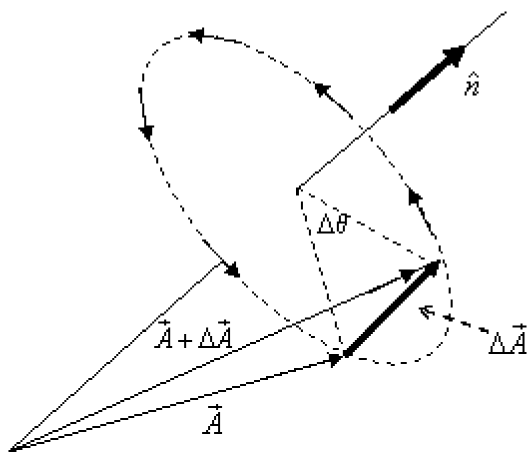
It is obvious that the vector $\Delta\vec{\theta} = \Delta\theta\hat{n}$. The corresponding derivative with respect time is called the angular velocity, usually denoted by $\vec{\omega}$. Thus

$$\vec{\omega} = \frac{d\theta}{dt}\hat{n}$$

I now point out that although the above equation is written for a position vector, there is nothing in its derivation that limits it to position vectors only. It is in fact true for any vector as can be easily proved by replacing the (*xyz*) coordinates by the corresponding components of the vector in the derivation above. Thus if a vector $\vec{A}$ is given an infinitesimal rotation $\Delta\vec{\theta} = \Delta\theta\hat{n}$, its will change by

$$\Delta\vec{A} = \Delta\vec{\theta} \times \vec{A}$$

This is shown pictorially in figure 6.



Change $\Delta\vec{A}$ in a vector $\vec{A}$ when it is rotated by $\Delta\theta$ about an axis in the direction of $\hat{n}$.

Figure 6

Let us now see how much does a vector $\vec{A}$ change when we apply two infinitesimal rotations $\Delta\vec{\theta}_1$ and $\Delta\vec{\theta}_2$ about two different axes. Let the vector be denoted by $\vec{A}_1$ after the first rotation and by $\vec{A}_2$ after the second one. Then we have

$$
\begin{aligned}
\vec{A}_1 &= \vec{A} + \Delta\vec{\theta}_1 \times \vec{A} \\
\vec{A}_2 &= \vec{A}_1 + \Delta\theta_2 \times \vec{A}_1 \\
&= \vec{A} + \Delta\vec{\theta}_1 \times \vec{A} + \Delta\vec{\theta}_2 \times \left(\vec{A} + \Delta\vec{\theta}_1 \times \vec{A}\right) \\
&= \vec{A} + \left(\Delta\vec{\theta}_1 + \Delta\vec{\theta}_2\right) \times \vec{A} \text{ (up to first order in } \Delta\theta)
\end{aligned}
$$

thereby showing that for several infinitesimal rotations the final effect can indeed be expressed by adding the effect of each one of them.

Next we consider the rate of change of a vector rotating with an angular velocity $\vec{\omega}$. It is obtained as follows:

$$
\Delta\vec{A} = \Delta\vec{\theta} \times \vec{A}
$$

$$
\frac{\Delta\vec{A}}{\Delta t} = \frac{\Delta\vec{\theta}}{\Delta t} \times \Delta\vec{A} \quad \Rightarrow \quad \frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}
$$

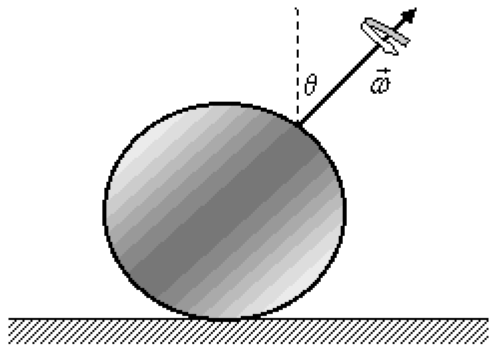
This is the rate of change of a vector $\vec{A}$ only due to its rotation. If it changes additionally due to some other causes, that has to be added to the above change separately. If we take the vector $\vec{A}$ to be the position vector $\vec{r}$, we get the formula

$$
\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}
$$

for linear velocity of a particle due to pure rotation of its position vector.

You may ask this point why is it that we want to take $\Delta\vec{\theta}, \vec{\omega}$ as vector quantities. The answer is that we in doing our calculations, we should know whether a quantity is a scalar or a vector or something else so that mathematical operations on it can be appropriately defined. For example, now that we know that $\vec{\omega}$ is a vector quantity, we can take its components and deal with them independently. Let me give you an example.

Example 1: A ball is given a spin at speed w and then put on a rough floor with $\vec{\omega}$ making an angle q with the vertical. When the ball eventually rolls, what would be its rolling speed (see figure 7)?



A spinning ball put on a rough floor with its angular velocity at an angle with the vertical.

Figure 7

In solving this problem, I make use of the vector nature of $\vec{\omega}$ and split it into its two components. It is the horizontal component $\omega \sin\theta$ that is responsible for making the ball roll. The vertical component $\omega \cos\theta$ does not contribute to rolling, as you well know. Further, this component eventually goes to zero due to friction. So the question is: if a sphere rotating with angular speed $\omega \sin\theta$ is kept on a rough floor with axis of rotation horizontal, what is its find rolling speed. I will let you figure that out. The point that is emphasized here is that knowing that $\vec{\omega}$ is a vector quantity helped us to solve the problem easily.

Now that we know $\vec{\omega}$ is a vector, the next question we ask is: how does $\vec{\omega}$ change when an external torque is applied on a body? So far we have learnt that external torques change angular momentum $\vec{L}$. So to know how $\vec{\omega}$ changes, we should know the relationship between $\vec{L}$ and $\vec{\omega}$. We derive this relationship next.

Angular momentum of a rigid body rotating with angular velocity $\vec{\omega}$: We now derive the relationship between the angular momentum of a body rotating in space with one point fixed. That means the body is not translating and has only three degrees of freedom. By definition, the angular momentum

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

For a rigid body rotating with one point fixed, I have derived above that $\vec{v}_i = \vec{\omega} \times \vec{r}_i$. With

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i = \left(\omega_y z_i - \omega_x y_i\right)\hat{i} + \left(\omega_z x_i - \omega_x z_i\right)\hat{j} + \left(\omega_x y_i - \omega_y x_i\right)\hat{k}$$

we get

$$
\begin{aligned}
\vec{L} &= \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \\
&= \sum_i m_i (x_i \hat{i} + y_i \hat{j} + z_i \hat{k}) \times \left\{ (\omega_y z_i - \omega_x y_i)\hat{i} + (\omega_z x_i - \omega_x z_i)\hat{j} + (\omega_x y_i - \omega_y x_i)\hat{k} \right\}
\end{aligned}
$$

This gives the three components of the angular momentum to be

$$
\begin{aligned}
L_x &= \left(\sum_i m_i (y_i^2 + z_i^2) \right) \omega_x - \left(\sum_i m_i x_i y_i \right) \omega_y - \left(\sum_i m_i x_i z_i \right) \omega_z \\
&= I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z
\end{aligned}
$$

$$
\begin{aligned}
L_y &= -\left(\sum_i m_i y_i x_i \right) \omega_x + \left(\sum_i m_i (z_i^2 + x_i^2) \right) \omega_y - \left(\sum_i m_i y_i z_i \right) \omega_z \\
&= I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z
\end{aligned}
$$

$$
\begin{aligned}
L_z &= -\left(\sum_i m_i z_i x_i \right) \omega_x - \left(\sum_i m_i z_i y_i \right) \omega_y + \left(\sum_i m_i (x_i^2 + y_i^2) \right) \omega_z \\
&= I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z
\end{aligned}
$$

This is usually written in the matrix form

$$
\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}
$$

The (3 x 3) matrix in the equation above is known as the moment of inertia tensor. Its diagonal terms

$$
I_{xx} = \sum_i m_i (y_i^2 + z_i^2) \quad I_{yy} = \sum_i m_i (z_i^2 + x_i^2) \quad I_{zz} = \sum_i m_i (x_i^2 + y_i^2)
$$

are the moments of inertia about the x , y and the z -axis, respectively. The off-diagonal terms

$$
I_{xy} = I_{yx} = -\sum_i m_i x_i y_i \quad I_{xz} = I_{zx} = -\sum_i m_i x_i z_i \quad I_{yz} = I_{zy} = -\sum_i m_i y_i z_i
$$

are known as the products of inertia. The values of the moments and products of inertia depend on the set of axes chosen.

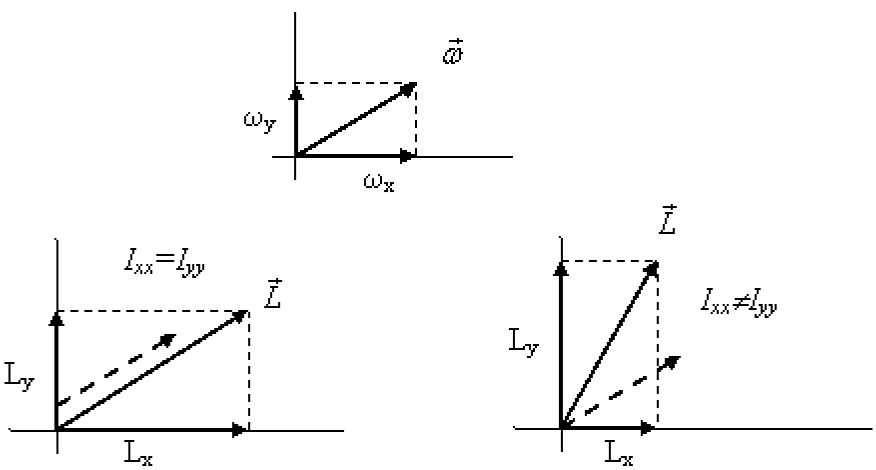
So you see that relationship between $\vec{L}$ and $\vec{\omega}$ is quite involved. Luckily, for a rigid body, for each point one can find a set of axes about so that products of inertia about that point vanish. These are known as the principal axes. Thus for the principal set of axes at a point

$$I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

These axes are attached with the body and rotate with it. However, the principal axes offer an advantage when dealing with the angular momentum of a rigid body. At a given time, if I calculate the components of the angular momentum by taking the rigid-body to be rotating in the principal axes frame at that instant, they turn out to be simply $L_x = I_{xx}\,\omega_x$, $L_y = I_{yy}\,\omega_y$ and $L_z = I_{zz}\,\omega_z$. Thus the angular momentum of the body is given as

$$\vec{L} = I_{xx}\omega_x\hat{i} + I_{yy}\omega_y\hat{j} + I_{zz}\omega_z\hat{k}$$

at any given instant. It is easily seen from the expression above that in general the angular momentum and the angular velocity are not parallel; they will be parallel only if $I_{xx} = I_{yy} = I_{zz}$, i.e. if all three moments of inertia about the principal axes are equal. This is shown in figure 8 in two dimensions.

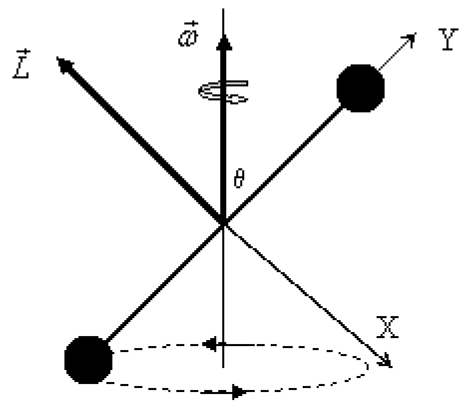


Direction of $\vec{L}$ for a given $\vec{\omega}$ (upper figure). If $I_{xx} = I_{yy}$, $\vec{L}$ and $\vec{\omega}$, shown by dashed arrow, are parallel (lower left). If $I_{xx} \neq I_{yy}$, $\vec{L}$ and $\vec{\omega}$ are not parallel (lower right).

Figure 8

Let me now solve an example.

Example 2: A thin massless rod of length *2l* has a point mass *m* at both its ends. It is rotating with angular speed w about a vertical axis passing through its centre and at an angle θ from it, as shown in figure 9. Calculate its angular momentum.



A thin massless rod with point mass m at both its ends rotating about a vertical axis. The principal axes are shown as X and Y axis with Z perpendicular to the XY plane.

Figure 9

We will apply the formula for angular momentum derived above. It is easy to see that at the centre of the rod, the principal axes are: one axis parallel to the rod and two of them perpendicular to it. These are shown in the figure above. Notice that the principal axes rotate with the body. The moment of inertia with respect to the principal axes shown in figure 9 are

$$I_{xx} = 2ml^2 \quad I_{yy} = 0 \quad I_{zz} = 2ml^2$$

The components of the angular velocity along the principal axes are

$$\omega_x = -\omega \sin\theta \quad \omega_y = \omega\cos\theta \quad \omega_z = 0$$

Thus the angular momentum is given as

$$\begin{aligned}\vec{L} &= I_{xx}\omega_x\hat{i} + I_{xy}\omega_y\hat{j} + I_{zz}\omega_z\hat{k} \\ &= -2ml^2\omega\sin\theta\hat{i}\end{aligned}$$

This is also shown in figure 9. It is clear from the figure that as the body rotates so does its angular momentum vector. Thus the angular momentum of the body changes with time although its magnitude remains unchanged.

I end this lecture by asking you to solve a similar problem.

Exercise: A rectangular thin sheet of sides *a* and *b* is rotating about one of its diagonals (see figure 10) with angular speed ω. The mass of the sheet is *m*. What is its angular momentum? Express it in terms of the principal axes unit vectors.

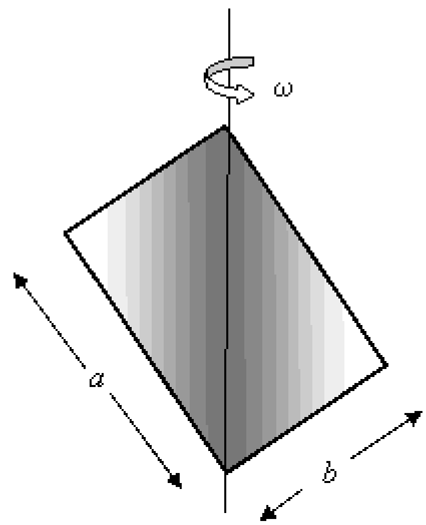


Figure 10

Lectures 22 & 23
Rotational dynamics V: Kinetic energy, angular momentum and torque in 3-dimensions

You learnt in the previous lecture is that the angular velocity $\vec{\omega}$ is a vector quantity pointing in the direction of the axis of rotation. Any vector that is rotating about $\vec{\omega}$ also changes direction. Thus the vector changes even if its magnitude is constant. If the vector is $\vec{A}$ then its rate of change purely on the basis of rotation is

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

Thus the velocity of a rotating particle at position $\vec{r}_i$ from the origin is

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

I also derived the general expression for the angular momentum, which is given as

$$\begin{aligned}\vec{L} &= (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\hat{i} \\ &\quad + (I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\hat{j} \\ &\quad + (I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\hat{k}\end{aligned}$$

Here I_{xx}, I_{yy} and I_{zz} are the moments of inertia about the *x*, *y* and the *z* axes, respectively. The off diagonal elements like I_{xy} are the products of inertia. A simplification in the expression above arises by employing the principal axes for which the products of inertia vanish. For convenience in writing, the principal axes are usually denoted by (*1,2,3*) instead of (*x,y,z*). Using this notation the angular momentum vector can be written in a simple form as

$$\vec{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}$$

where ω_1 , ω_2 and ω_3 are the components of the angular velocity along the principal axes. I now derive the expression for kinetic energy for a rigid body rotating with one point fixed.

Kinetic energy of a rotating rigid body: I consider a rigid body rotating with angular velocity $\vec{\omega}$. Its kinetic energy *T* is calculated as follows

$$T = \frac{1}{2}\sum_i m_i \vec{v}_i \cdot \vec{v}_i$$

Substituting $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ for of the velocities above and making use of some identities of vector products we get

$$\begin{aligned}T &= \frac{1}{2}\sum_i m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i) \\ &= \frac{1}{2}\sum_i m_i \vec{r}_i \cdot (\vec{v}_i \times \vec{\omega}) \\ &= \frac{1}{2}\sum_i m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) \\ &= \frac{1}{2}\vec{\omega}.\vec{L}\end{aligned}$$

In the principal axes therefore

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

This is the expression for the kinetic energy in terms of the principal moments of inertia and the components of angular velocity along the principal set of axes. Having obtained the general expressions for the angular momentum and kinetic energy of a rigid body, we now study the dynamics of a rigid body through the angular-momentum torque equation. Along the way I will explain the three observations that I had started my previous lecture with.

Dynamics of a rigid body: Dynamics of a rigid body is governed by the equation

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{applied}$$

and it is this equation that governs everything about the rigid-body rotation. What makes the motion of a rigid-body interesting is that there is a fantastic interplay between the angular momentum, angular velocity of a rigid body with or without an applied torque. For example if the angular velocity and the angular momentum of a rigid body are not parallel, the $\vec{L}$ vector would rotate about $\vec{\omega}$ and that would make $\vec{L}$ change. However, if there is no torque applied on the body, angular momentum cannot change. Therefore to compensate the change in $\vec{L}$ arising from its rotation, the angular velocity $\vec{\omega}$ itself must change. Changing $\vec{\omega}$ would make body rotate in a different way and this goes on. It is thus this interplay between $\vec{L}$ and $\vec{\omega}$ that makes a rigid body move in seemingly counterintuitive ways.

As a body rotates, its angular momentum changes on two counts: first because in general $\vec{L}$ and $\vec{\omega}$ are not parallel and therefore $\vec{L}$ rotates about $\vec{\omega}$. With

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

and

$$\vec{L} = I_1 \omega_1 \hat{i} + I_2 \omega_2 \hat{j} + I_3 \omega_3 \hat{k}$$

the rate of change of $\vec{L}$ only due to its rotation about $\vec{\omega}$ is given as

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \vec{\omega} \times \vec{L} \\ &= (\omega_2 L_3 - \omega_3 L_2)\hat{i} + (\omega_3 L_1 - \omega_1 L_3)\hat{j} + (\omega_1 L_2 - \omega_2 L_1)\hat{k} \end{aligned}$$

If the components ω_1 , ω_2 and ω_3 were also changing, I would have to add an additional term on the right-hand side of the expression above to take care of that. This is the second reason for the change in angular momentum of the body. For the time being I focus on cases where the components of $\vec{\omega}$ along the principal axis remain unchanged. This in turn implies that the magnitude of the angular momentum remains constant during the rotational motion of the body. This happens when the applied torque is always perpendicular to the angular momentum. Substituting for L_1 , L_2 and L_3 in the equation above, I get

$$\frac{d\vec{L}}{dt} = \omega_2\omega_3(I_3 - I_2)\hat{i} + \omega_3\omega_1(I_1 - I_3)\hat{j} + \omega_1\omega_2(I_2 - I_1)\hat{k}$$

So at any instant the components of $\frac{d\vec{L}}{dt}$ are

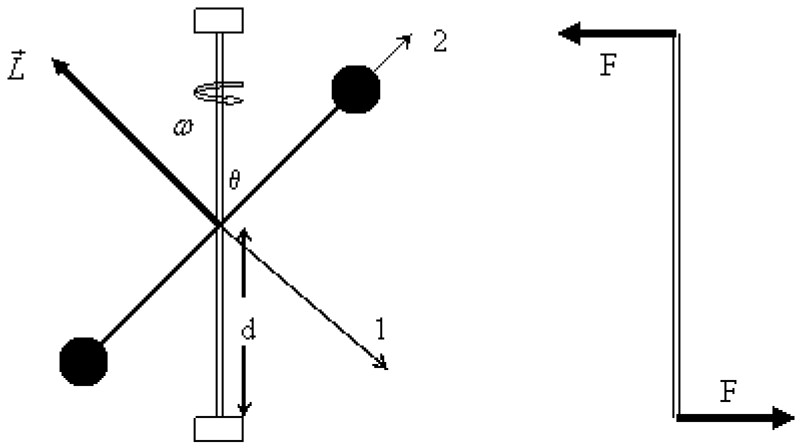
$$\left(\frac{d\vec{L}}{dt}\right)_1 = \omega_2\omega_3(I_3 - I_2) \quad \left(\frac{d\vec{L}}{dt}\right)_2 = \omega_3\omega_1(I_1 - I_3) \quad \text{and} \quad \left(\frac{d\vec{L}}{dt}\right)_3 = \omega_1\omega_2(I_2 - I_1)$$

For a geometric interpretation of these equations I urge you to go back to the previous lecture and see how we obtained the changes in the coordinates of the end of a rod rotating infinitesimally. This gives the components of the torque required to be

$$\tau_1 = \omega_2\omega_3(I_3 - I_2) \qquad \tau_2 = \omega_3\omega_1(I_1 - I_3)$$
$$\tau_3 = \omega_1\omega_2(I_2 - I_1)$$

To apply these equations I start with calculation of torque for the example that we solved at the end of the previous lecture.

Example 1: A thin massless rod of length *2l* has a point mass *m* at both its ends. It is rotating with angular speed w about a vertical axis passing through its centre and at an angle θ from it, as shown in figure 1. If the axis of rotation is held at its two ends by ball bearings, calculate the force that the ball bearings apply on the axis. The ball bearings are placed symmetrically from the centre of the rod at a distance *d* each.



A thin massless rod with point mass m at both its ends rotating about a vertical axis (left). The axis of rotation is kept fixed in place by two ball bearings at a distance of d from the centre of the rod. The forces on the rod applied by bearings are also shown (right).

Figure 1

Recall from the previous lecture that I had taken the principal axes (*1*,*2*,*3*) with (*1*,*2*) as shown in figure 1 and axis *3* perpendicular to them. The moments of inertia about the principal axes are

$$I_1 = 2ml^2 \quad I_2 = 0 \quad \text{and} \quad I_3 = 2ml^2$$

The angular velocity and the angular momentum of the rod-mass system are

$$\begin{aligned}\vec{\omega} &= \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k} \\ &= -\omega\sin\theta\hat{i} + \omega\cos\theta\hat{j}\end{aligned}$$

and

$$\begin{aligned}\vec{L} &= I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k} \\ &= -2ml^2\omega\sin\theta\hat{i}\end{aligned}$$

All the parameters - mass m , length l and angle θ - in the equation above are constant so the magnitude of the angular momentum is also a constant. As such we can apply the formulae given above to get the components of the torque to be applied as

$$\tau_1 = \omega_2\omega_3(I_3 - I_2) = 0 \qquad \tau_2 = \omega_3\omega_1(I_1 - I_3) = 0$$
$$\tau_3 = \omega_1\omega_2(I_2 - I_1) = 2ml^2\omega^2\sin\theta\cos\theta$$

Thus the torque needed to keep the rotating rod in its position is in the direction of principal axis 3 of the body. As was noted above, the torque is indeed perpendicular to $\vec{L}$. The torque is provided by the forces applied by the bearings. When the rod is in the plane of the paper, as shown in the figure, the force would be to the left at the upper end and to the right at the lower end of the rod (see figure 1). And their magnitudes will be equal since the CM of the rod has zero acceleration. Thus the forces provide a couple equal to τ_3. Their magnitude is

$$F = \frac{2ml^2\omega^2 \sin\theta\cos\theta}{2d} = \frac{ml^2\omega^2 \sin\theta\cos\theta}{d}$$

There is another method of calculating $\frac{d\vec{L}}{dt}$ that we describe now. $\vec{L}$ has one component $L_V = 2ml^2\omega\sin^2\theta$ in the direction of $\vec{\omega}$ and the other component $L_H = 2ml^2 \sin\theta\cos\theta$ perpendicular to $\vec{\omega}$ (see figure 2).

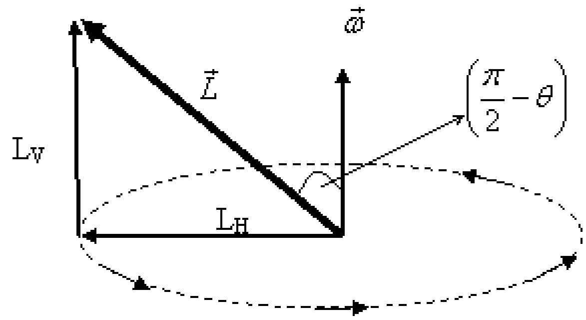


Figure 2

As the rod rotates L_V remains unchanged but L_H sweeps a circle with angular frequency ω. The rate of change of $\vec{L}$ is therefore the same as that of L_H. The magnitude of the latter is ωL_H. Since at the position shown, the tip of L_H is moving out of the paper, the direction of the change in L_H is also the same. This is the direction of principal axis 3. It thus follows that

$$\left|\frac{d\vec{L}}{dt}\right| = \omega L_H$$

$$= 2ml^2\omega^2 \sin\theta\cos\theta$$

in the direction of principal axis 3. For completeness I also calculate the kinetic energy of the rod-mass system. It is

$$T = \frac{1}{2}I_1\omega_1^{\ 2} + \frac{1}{2}I_2\omega_2^{\ 2} + \frac{1}{2}I_3\omega_2^{\ 2}$$
$$= ml^2\omega^2 \sin^2\theta$$

I now give you a couple of exercises similar to the problem above.

Exercise 1: In the problem above, if the axis of rotation passes through a different point than the centre of the rod (see figure 3), what will be the forces applied by the bearings with everything else remaining the same? (***Hint:*** *the CM is now moving in a circle*)

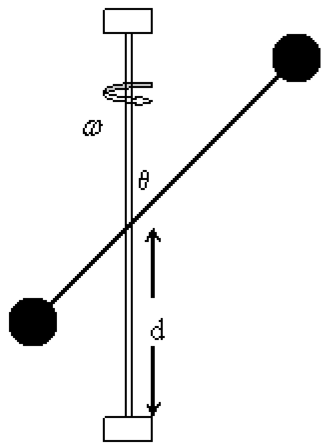
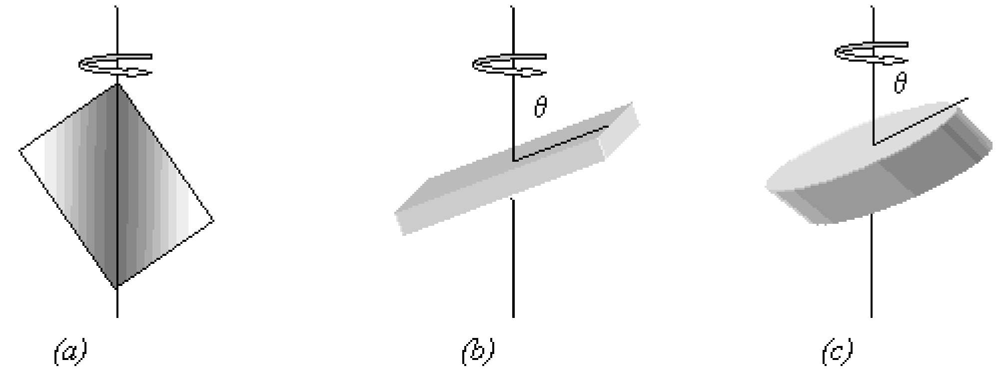


Figure 3

Exercise 2: For the rotating objects shown below in figure 4, calculate the rate of change of their angular momentum by the two methods employed in the example above.

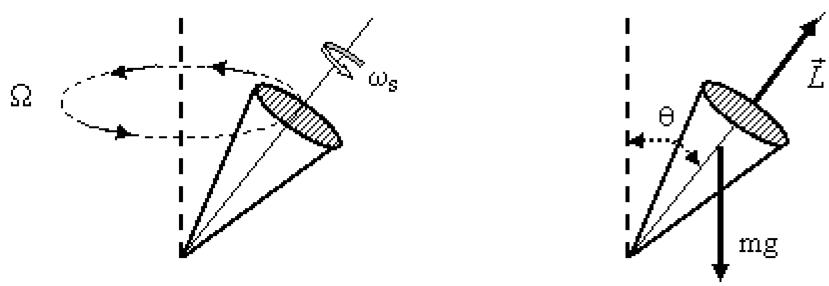


Find the rate of change of angular momentum of (a) a rectangular sheet rotating about its diagonal, (b) a rectangular sheet rotating about an axis passing through its centre, and (c) a thin disc rotating about an axis passing through its centre.

Figure 4

If you have followed the example above, and have also done the exercises suggested, then you will be in a position to understand the explanation of two of the three observations I started my previous lecture with. The two observations were the precession of a spinning top and only one roller of the three shown being able to go over a curved track entirely.

Example 2: Let me take the case of the precession of a spinning top. In this case we observe that when a spinning top is put on a floor and its lower point is held at one point, it starts precessing about the vertical axis (see figure 5)

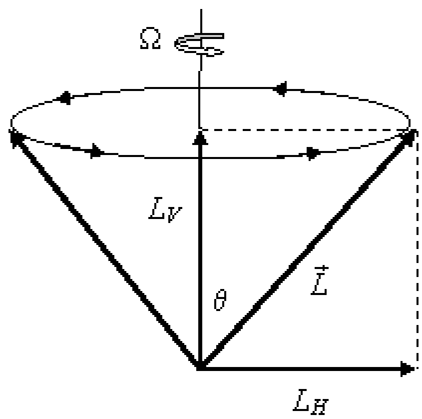


A precessing top (left) and its angular momentum and force mg acting on it (right).

Figure 5

I take the mass of the top to be *m*, its moment of inertia about the spinning axis I , distance of its CM from the pivot point l and its spinning rate to be ω_s. The top's axis is making an angle θ from the vertical. Let us take the rate of precession, i.e. the angular speed at which the top starts to rotate about the vertical to be Ω. It is observed that Ω is usually much smaller than ω_s. So in calculating angular momentum we are going to take it as arising from the spin only and neglect any contribution of Ω to it. The angular momentum is then along the spin axis of the top and its magnitude is $L = I\omega_s$, where I is top's moment of inertia about its axis. Further, there is torque acting on the top due to its weight. The magnitude of the torque is $mgl\sin\theta$ and it is perpendicular to the plane formed by the vertical and the spin axis (the direction of $\vec{L}$). At the position shown in figure 5, the torque is going into the plane of the paper. The problem then reduces to the following. A rigid body has an angular momentum $\vec{L}$ and is being acted upon by a torque of magnitude $mgl\sin\theta$ perpendicular to $\vec{L}$. What will happen to the body?

Since the angular momentum is being acted upon by a torque perpendicular to it, it changes continuously with time with its magnitude remaining unaffected. Thus it moves on the surface of a cone as shown in figure 6.



Angular momentum vector moving on the surface of a cone at a constant rate. Its vertical and horizontal components are L_V and L_H, respectively.

Figure 6

Let me now calculate the frequency of rotation of vector $\vec{L}$. For this I again look at the vertical L_V and horizontal L_H components of the angular momentum, as shown in figure 6. The vertical component remains unchanged and the horizontal component changes at the rate ΩL_H as the $\vec{L}$ vector rotates. This gives $\left|\left(\frac{dL}{dt}\right)\right| = \Omega L\sin\theta$, which should be equal to the torque. Substituting $L = I\omega_s$, I thus get

$$\Omega I\omega_s \sin\theta = mgl\sin\theta$$

$$\Omega = \frac{mgl}{I\omega_s}$$

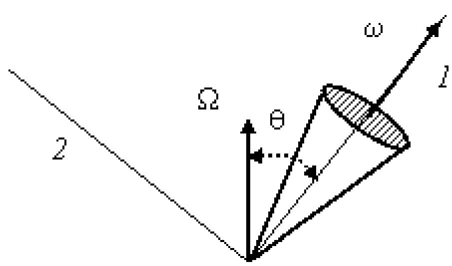
This is the rate at which the $\vec{L}$ vector rotates. Since $\vec{L}$ is attached to the top, the top also rotates at the same rate. $\Omega = \dfrac{mgl}{I\omega_s}$ is then the rate of precession of the cone.

As the top precesses, its CM moves in a circle. You may now wonder where does the centripetal force for this come from? This is provided by the horizontal reaction or the frictional force at the pivot point. Second question you may raise is why is it that the component L_H starts moving in a horizontal circle due to the torque while the vertical component does not move in a vertical circle. In the actual motion, it does. So in addition to the precession, the axis of the top also oscillates up and down with very small amplitude. If you are careful in you observations, you will see this motion. This is known as the nutation of the top. In our present treatment, we have ignored this motion and solved the problem only to get the precession rate.

I now wish to explore if to get this answer, I could equivalently have used the equations

$$\tau_1 = \omega_2\omega_3(I_3 - I_2)$$
$$\tau_2 = \omega_3\omega_1(I_1 - I_3)$$
$$\tau_3 = \omega_1\omega_2(I_2 - I_1)$$

To do this, let me first identify the principal axes of the cone at the pivot point and label them. The principal axes are the spin axis and two other axes perpendicular to it. These are shown and labeled *(1,2,3)* in figure 7; in this position axes 1 and 2 are in the plane of the paper and axis 3 is coming out of it.



A spinning top precessing about the vertical. Its principal axes 1 and 2 are shown.

Figure 7

The moments of inertia about the principal axes are $I_1 = I, I_2 = I_3 = I_\perp$. The components of $\vec{\omega}$ at the instant (I take it to be time $t = 0$) shown in figure 7 are

$$\omega_1 = \omega_s + \Omega\cos\theta \approx \omega_s \qquad \omega_2 = \Omega\sin\theta$$
$$\omega_3 = 0$$

Substituting the values of moments of inertia and the angular velocity components in the equations for the components of the torque gives

$$\tau_1 = 0 \quad \tau_2 = 0$$
$$\tau_3 = \Omega\omega_s \sin\theta(I_\perp - I)$$

This is not the same answer as obtained earlier. Where have we gone wrong? Is the previous answer correct or is this answer correct? We will see later that in applying the equations above, we have not taken into account the fact that due to the spin of the top, its principal axes also spin about axis *1* and that makes the components of $\vec{\omega}$ along them time-dependent. For now I move on to explain the observation about only one of the rollers being able to go over all the curves of a track.

Example 3: If you have performed the experiment, you would have seen that only roller 1 (see figure 8) that is tapering down as we move away from its centre is able to go over all the curves. Let me now explain that.

As a roller goes over a curve, its centre of mass moves requires a centripetal force to do so. At the same time, the angular momentum of the roller also changes direction and that requires a torque. Both the centripetal force and the torque are provided by the normal reaction of the track on the rollers. These reaction forces on the three rollers are shown in figure 9.

In analyzing the motion of these rollers, I am taking them to be moving into the paper. Thus the direction of their angular momentum is to the left, as shown in the figure. Now if these rollers have to make a turn, the normal reactions should provide the required centripetal force in the horizontal direction. This rules out the plain cylindrical roller (roller 2) from making any turn because both normal reactions on it are in the vertical direction. This leaves the other two cylinders for further consideration. For those rollers, the torque of the normal reaction forces about the CM should change their angular momentum vector in the appropriate direction. Let us look at roller *1* first.

Roller 1: For a left turn, $N_1 < N_2$ for centripetal force. Therefore the torque generated by them is in the direction coming out of the page. As the roller makes a left turn, the associated change in its angular momentum also is in the direction coming out of the page, consistent with the torque generate. For a right turn by this roller, the centripetal force is to the right so $N_1 > N_2$. This generates a torque about the CM that goes into the page. For the right turn, the change in the angular momentum is also into the page, consistent with the torque generated. Thus for roller *1* , the centripetal force and the torque generated are consistent with the centripetal force and the change in its angular momentum. Let us now see what happens to roller *3* .

Roller 3: If roller 3 turns left, the centripetal force will be provided correctly if $N_1 > N_2$. This however gives a torque about the CM that is going into the page. On the other hand, during left turn the change in the angular momentum comes out of the page. Thus the torque and the change in angular momentum are in opposite directions. Exactly the same situation arises for a right turn. Because of this inconsistency, the roller fails to turn at any of the curves. This example teaches us about the centre of mass motion combined with angular momentum changes about the CM. We now move on to discuss the general form of the equation relating the torque and the angular momentum.

The general equation governing rotation of a rigid body: Having dealt with situations where components of $\vec{\omega}$ are constant, we now ask what happens when $\vec{\omega}$ is also changed. For this let me look at the expression for the angular momentum in the principal axis frame again. It is

$$\vec{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}$$

I now give a slightly different derivation for the rate of change of $\vec{L}$. In doing this derivation I keep in mind that as a rigid body rotates, the unit vectors along its principal axes also rotate and their rate of change is (see previous lecture)

$$\frac{d\hat{i}}{dt} = \vec{\omega}\times\hat{i} \qquad \frac{d\hat{j}}{dt} = \vec{\omega}\times\hat{j} \qquad \frac{d\hat{k}}{dt} = \vec{\omega}\times\hat{k}$$

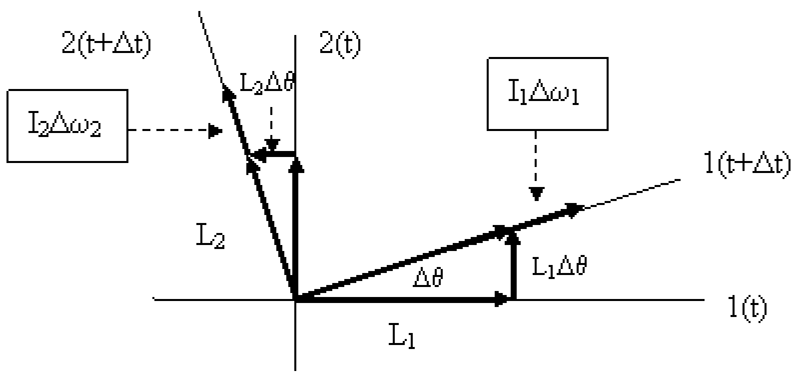
Now I differentiate $\vec{L}$ to get

$$\frac{d\vec{L}}{dt} = \left(I_1\frac{d\omega_1}{dt}\hat{i} + I_2\frac{d\omega_2}{dt}\hat{j} + I_3\frac{d\omega_3}{dt}\hat{k}\right) + \left(I_1\omega_1\frac{d\hat{i}}{dt} + I_2\omega_2\frac{d\hat{j}}{dt} + I_3\omega_3\frac{d\hat{k}}{dt}\right)$$

$$= \left(I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k}\right) + \vec{\omega}\times\left(I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}\right)$$

$$= \left(I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k}\right) + \vec{\omega}\times\vec{L}$$

Here the first term is due to the change in the components of $\vec{\omega}$ along the principal axis and the second term is the change in $\vec{L}$ due to its rotation. Notice that we recover the formula derived earlier if the components of $\vec{\omega}$ do not change with time, i.e. $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. Let me repeat the interpretation of the equation: at any instant we take the body rotating in the principal axes frame at that time, i.e. the frame is frozen at its position at that time and the body is taken to be rotating in it. To see this geometrically, let me take a two-dimensional case. Shown in figure 10 are the principal axes 1 and 2 of a rigid body at times *t* and *(t+ Δt)* . In time interval *Δt* the body and the frame attached to it rotate by an angle $\Delta\vec{\theta} = \Delta\theta\hat{k}$, and ω_1 and ω_2 change to $\omega_1 + \Delta\omega_1$ and $\omega_2 + \Delta\omega_2$. With these changes let me calculate changes in the components L_1 and L_2 in the frame frozen at time *t* .



The principal axes of a rotating body at time t and (t+Δt). In time interval Δt the body and the attached frame have rotated by an angle Δθ about the z-axis and the components of angular velocity have changed by $\Delta\omega_1$ and $\Delta\omega_2$.

Figure 10

Looking at the figure, where I have shown all the changes that have taken place during the time interval *Δt* , we get in the frame at time *t*

$$\begin{aligned}\Delta L_1 &= I_1\Delta\omega_1\cos\Delta\theta - L_2\Delta\theta \\ &= I_1\Delta\omega_1 - L_2\Delta\theta\end{aligned}$$

and

$$\begin{aligned}\Delta L_2 &= I_2\Delta\omega_2\cos\Delta\theta + L_1\Delta\theta \\ &= I_2\Delta\omega_2 + L_1\Delta\theta\end{aligned}$$

So the total change in the angular momentum is

$$\begin{aligned}\Delta\vec{L} &= I_1\Delta\omega_1\hat{i} + I_2\Delta\omega_2\hat{j} - L_2\Delta\theta\hat{i} + L_1\Delta\theta\hat{j} \\ &= I_1\Delta\omega_1\hat{i} + I_2\Delta\omega_2\hat{j} + \Delta\theta\hat{k}\times\left(L_1\hat{i} + L_2\hat{j}\right)\end{aligned}$$

Dividing both sides by Δt and taking proper limit gives

$$\frac{d\vec{L}}{dt} = I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + \vec{\omega}\times\vec{L}$$

This gives you some idea about where this equation comes from. Of course in a more accurate treatment, rotations about the other axes also have to be taken into account. For infinitesimal rotations, they can all be added up and give the general equation

$$\frac{d\vec{L}}{dt} = \left(I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k}\right) + \vec{\omega}\times\vec{L}$$

This gives

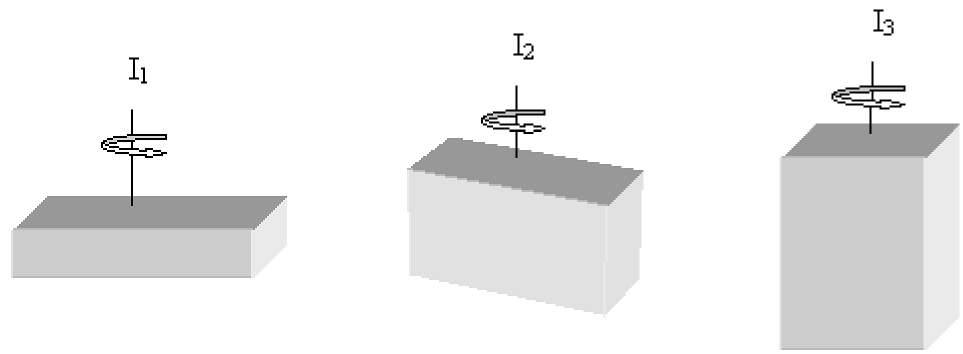
$$\begin{aligned}
\dot{L}_1 &= I_1\dot{\omega}_1 + (\vec{\omega}\times\vec{L})_1 \\
&= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\
\dot{L}_2 &= I_2\dot{\omega}_2 + (\vec{\omega}\times\vec{L})_2 \\
&= I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) \\
\dot{L}_3 &= I_3\dot{\omega}_3 + (\vec{\omega}\times\vec{L})_3 \\
&= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)
\end{aligned}$$

Each one of these rates of change should be equal to the component of the torque in that direction .Thus

$$\begin{aligned}
\tau_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\
\tau_2 &= I_2\dot{\omega}_2 + \omega_2\omega_3(I_1 - I_2) \\
\tau_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)
\end{aligned}$$

These are the most general equations governing the dynamics of a rigid body and are known as Euler's equations. I now use it to explain the third experiment I had suggested in the beginning of Lecture 21.

Example 4: Hold a rectangular box at a height with one of its faces perpendicular to the vertical, give it a spin and let it drop (see figure 11). Describe its subsequent rotational motion.



A box being given spin about different axes and being dropped from a height. Its moments of inertia about three different axes are shown.

Figure 11

This is an example of torque-free ($\vec{\tau} = 0$) motion because there is no torque on the box about its centre of mass. Thus its rotational motion is governed by the equations

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = 0$$
$$I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_2) = 0$$
$$I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = 0$$

For a box similar to the one shown in figure 11 we would generally have $I_3 > I_2 > I_1$.

Let me first consider the case when the box is given a spin about its principal axis 1. Let me also assume that in the process I also disturb it and give it very small angular velocities ω_2 and ω_3 about its axes 2 and 3, respectively. Since both ω_2 and ω_3 are very small, their product is second-order in smallness and will be ignored. The Euler equations and there are then as given below.

$$I_1 \dot{\omega}_1 = 0 \qquad \cdots (I)$$
$$I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) = 0 \qquad \cdots (II)$$
$$I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = 0 \qquad \cdots (III)$$

The first equation implies that ω_1 is a constant. Let me call it the spin rate ω_0 . Using this fact the other two equations are dealt with as follows. Differentiate equation (II) with respect to time to get

$$I_2 \ddot{\omega}_2 + \omega_0 \dot{\omega}_3 (I_1 - I_3) = 0$$

and substitute for $\dot{\omega}_3$ from equation (III) to obtain

$$\ddot{\omega}_2 - \frac{(I_1 - I_3)(I_2 - I_1)\omega_0^2}{I_2 I_3}\omega_2 = 0$$

Since $I_3 > I_2 > I_1$, the equation above is of the form

$$\ddot{\omega}_2 + \Omega^2\omega_2 = 0 \quad \text{where} \quad \Omega^2 = \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\omega_0^2$$

Its solution is of the form

$$\omega_2 = A\sin\Omega t + B\cos\Omega t$$

One can similarly get equation for ω_3 also and see that it also has similar oscillatory solution. This implies that as the box falls down it spins about axis 1 and oscillates about axes 2 and 3. Since magnitudes of ω_2 and ω_3 are small, you see the box fall essentially spinning only. The same thing will happen if we give initial spin about axis 3. However something different happens when the initial spin is about axis 2. Assuming ω_1 and ω_3 to be small, in this case the Euler equations take the following form right after the release of the box.

$$\ddot{\omega}_1 - \frac{(I_2 - I_1)(I_3 - I_2)\omega_0^2}{I_1 I_3}\omega_1 = 0$$
$$\ddot{\omega}_2 = 0$$
$$\ddot{\omega}_3 - \frac{(I_2 - I_1)(I_3 - I_2)\omega_0^2}{I_1 I_3}\omega_3 = 0$$

The second equation above implies that ω_2 is a constant and with $I_3 > I_2 > I_1$, the other two equations take the form

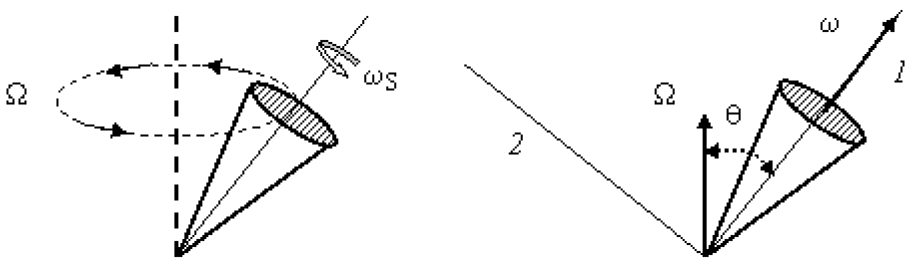
$$\ddot{\omega}_2 - \Omega^2\omega_2 = 0\,, \quad \ddot{\omega}_3 - \Omega^2\omega_3 = 0 \quad \text{where} \quad \Omega^2 = \frac{(I_2 - I_1)(I_3 - I_2)}{I_2 I_3}\omega_0^2$$

Solution of these equations is of the form

$$A\exp(\Omega t) + B\exp(-\Omega t)$$

which indicates that right after the release, the angular velocities about axes 1 and 3 will grow very fast and take on a large value. Thus the box will start rotating about all three axes and that is what you observe. Thus we see that a rigid body is stable when it is given a spin about the axes having the smallest or the largest moment of inertia. However, if given a spin about the axis with intermediate moment of inertia, it will be unstable. Next I take up the case of precessing top that I had not solved by employing Euler's equations earlier. This is an example where a torque is also being applied on the system

Example 5: Apply Euler's equations to a precessing top and get its precession frequency Ω. The top has a mass *m* and is spinning at a rate of ω_S (see figure 12). Its centre of gravity is at a distance *l* from the pivot point.



A precessing top (left) and its principal axes and spin and precession angular velocities at time t=0 (right).

Figure 12

I have already discussed about the principal axes of the top in example 2 above. With $I_1 = I, I_2 = I_3 = I_\perp$ the Euler's equations for the top are

$$\tau_1 = I_1\dot{\omega}_1$$
$$\tau_2 = I_\perp\dot{\omega}_2 + \omega_3\omega_1(I - I_\perp)$$
$$\tau_3 = I_\perp\dot{\omega}_3 + \omega_1\omega_2(I_\perp - I)$$

Now in applying Euler's equations you have to keep in mind that the top is spinning. As such its principle axes 2 and 3 also rotate about axis 1 with angular frequency ω_S . So the components of angular frequency and torque in the direction of these axes also change with time. Taking time at which the position of the top is shown in figure 12 to be $t = 0$, I draw in figure 13 the position of axes 2 and 3 at time *t* . In this figure, I have neglected the angle W t through which the top and therefore the torque vector itself has rotated. In other words I have assumed that $\omega_S >> \Omega$. Thus the angular velocity and torque are shown where they were at $t = 0$.

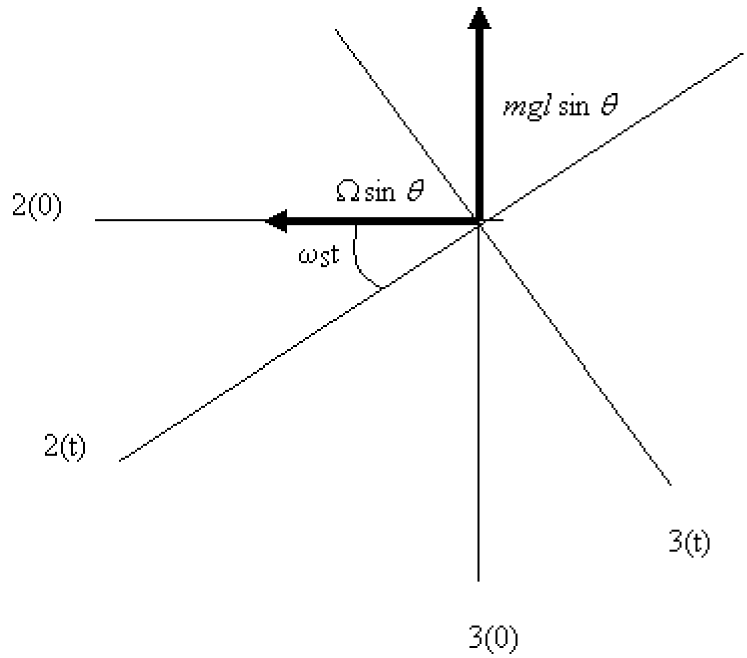


Figure 13

Looking at figure 13, it is clear that the components of the angular velocity and the torque are

$$\omega_1 = \omega_S + \Omega\cos\theta \approx \omega_S \qquad \omega_2 = \Omega\sin\theta\cos(\omega_S t) \qquad \omega_3 = -\Omega\sin\theta\sin(\omega_S t)$$

$$\tau_1 = 0 \qquad \tau_2 = -mgl\sin\theta\sin(\omega_S t) \qquad \tau_3 = -mgl\sin\theta\cos(\omega_S t)$$

Substituting these in the Euler's equation for the top gives

$$\begin{aligned}
\dot{\omega}_1 &= 0 \\
-mgl\sin\theta\sin(\omega_S t) &= -I_\perp\Omega\omega_S\sin\theta\sin(\omega_S t) - (I - I_\perp)\Omega\omega_S\sin\theta\sin(\omega_S t) \\
-mgl\sin\theta\cos(\omega_S t) &= -I_\perp\Omega\omega_S\sin\theta\cos(\omega_S t) + (I_\perp - I)\Omega\omega_S\sin\theta\cos(\omega_S t)
\end{aligned}$$

The first of these equations gives $\omega_1 = constant = \omega_S$. The other two equations give the same answer which is

$$\Omega = \frac{mgl}{I\omega_S}$$

This is the answer that we have seen earlier. In solving the Euler's equations for the top, we made the assumption of $\omega_S >> \Omega$. Further we assumed that the top only precesses about the vertical. However, there is no reason why it cannot posses a horizontal angular velocity Ω_H also. Assuming the existence of Ω and Ω_H and then solving the Euler's equations will give a more complete solution for the motion of a spinning top. It in fact gives the nutating motion also. You may want to try getting this general solution.

With this lecture I end of the topic of rigid-body rotation.

Lecture 24
Harmonic oscillator I: Introduction

Having analyzed the motion of particles in different situations, let us now focus on a very special kind of motion: that of oscillations. This is a very general kind of motion seen around you: A partial moving around the bottom of a cup, a pendulum swinging, a clamped rod vibrating about its equilibrium position or a string vibrating. A good first approximation to these motions is the simple harmonic oscillation. Let us see what does that mean? At a stable equilibrium point, the force on a body is zero; not only that, as a particle moves away from equilibrium, its potential energy increases and it is pulled back towards the equilibrium point. Thus around a stable equilibrium point x_0 (for simplicity, let me take one-dimensional motion) the potential energy $\phi(x)$ can be written as

$$\phi(x_0+\Delta x)=\phi(x_0)+\left.\frac{d\phi}{dx}\right|_{x0}\Delta x+\frac{1}{2}\left.\frac{d^2\phi}{dx^2}\right|_{x0}\Delta x^2+.\cdots$$

Since at an equilibrium point, the force $F(x_0\,)$ on the particle vanishes,

$$\left.\frac{d\phi}{dx}\right|_{x0}=-F(x_0)=0$$

Further, because $\Phi(x)$ has a minimum at x_0 , this gives

$$\phi(x_0+\Delta x)=\phi(x_0)+\frac{1}{2}k\Delta x^2 \quad \text{with} \quad k=\left.\frac{d^2\phi}{dx^2}\right|_{x0}>0$$

Writing $\Delta x=y$ I get

$$\phi(y)=\phi(x_0)+\frac{1}{2}ky^2$$

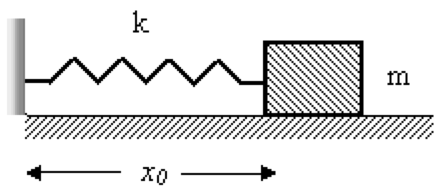
and the corresponding equation of motion for a mass *m* as

$$m\ddot{y}=-ky \quad \text{or} \quad m\ddot{y}+ky=0$$

As I will show a little later, the solution of this equation is of the form

$$y(t) = A\sin \omega t + B\cos \omega t$$

and is known as the simple harmonic motion. It is the simplest possible motion about a stable equilibrium point. Of course if $k = 0$, the force will have higher order dependence on y and the motion becomes more complicated. Further, even if $k \neq 0$, if we include higher order terms, the resulting motion will become more complex. It is for this reason that we call the motion above simple harmonic motion. We will see that this itself is quite a rich system. A system that performs simple harmonic motion is called a simple harmonic oscillator. A prototype if this system is the spring-mass system with k being the spring constant and m the mass of the block on the spring (figure 1).



A spring-mass system

Figure 1

In these lectures, I will talk about the motion of this system and how it is represented by a phasor diagram. I will then introduce damping into the system. The simplest damping is a constant opposing force like friction and next level is a damping proportional to the velocity. Then I will apply a force on the system and see the motion of force damped and undamped oscillator. Along the way, I will solve many examples to show wide applicability of simple harmonic motion.

To start with let us take our prototype system of mass and spring with unstretched length of the spring x_0 so that equilibrium distance of the mass is x_0. Now when the mass is displaced about x_0 by x in the positive direction, the force is in negative direction so that

$$m\ddot{x} = -kx$$

or

$$\ddot{x} + \omega_0^2 x = 0 \quad \text{with} \quad \omega_0^2 = \frac{k}{m}$$

This is the general equation for simple harmonic oscillator. Recall that in such cases we assume a solution of the form

$x = e^{\lambda t}$

and substitute it in the equation to get

$$\lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0$$

Since this equation is true for all times, we should have

$$\lambda^2 + \omega_0^2 = 0 \quad \Rightarrow \quad \lambda = \pm \hat{i}\,\omega_0$$

Thus there are two solution $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$. A general solution is then given in terms of a linear combination of the two solutions so let us write

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

Since $x(t)$ is real it is clear that $B = A^*$. Thus

$$\boxed{x(t) = Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t}}$$

If we take $A = A_R + iA_I$, where both A_R and A_I are real then the solution above takes the form

$$x(t) = 2A_R \cos \omega_0 t - 2A_I \sin \omega_0 t$$

which alternatively can be written as

$$\boxed{C \cos \omega_0 t + D \sin \omega_0 t}$$

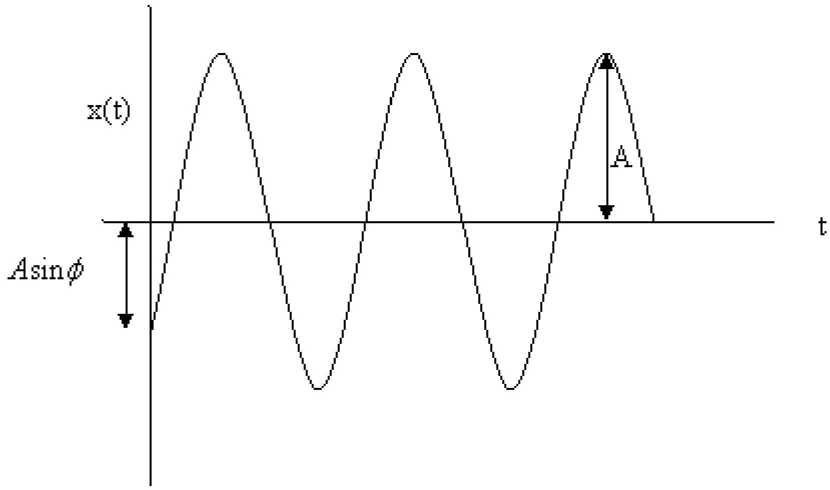
Another equivalent way of writing the solution is

$\boxed{x(t) = A \sin(\omega_0 t + \phi)}$ or $\boxed{x(t) = A \cos(\omega_0 t + \phi)}$

where

$$A = \sqrt{C^2 + D^2}\ ; \sin \phi = \frac{C}{\sqrt{C^2 + D^2}} \text{ and } \cos \phi = \frac{D}{\sqrt{C^2 + D^2}}$$

A is the maximum distance that the mass travels during a simple harmonic oscillation. It is known as the amplitude of oscillation. The quantity $(\omega_0 t + \phi)$ is known as the phase with Φ being the initial phase. All the boxed equations above are equivalent ways of writing the solution for a harmonic oscillator. The general graph depicting the solution $x(t) = A\sin(\omega_0 t + \phi)$ is given in figure 2.



General displacement of the mass in a spring-mass system

Figure 2

Thus A is the maximum distance traveled by the block and $A\sin\phi$ gives its initial displacement. The constants C and D or A and ϕ are determined by the initial conditions, i.e. initial displacement and velocity of the mass. In general any two conditions are enough to determine the constants.

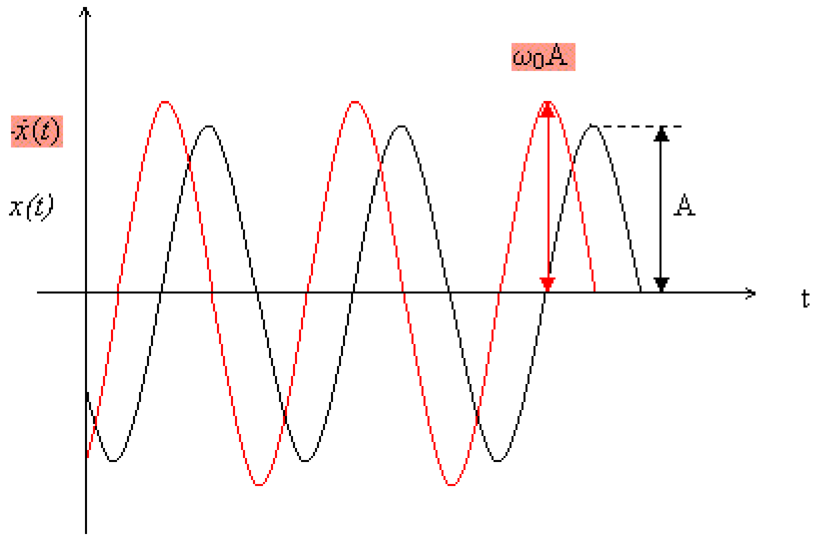
For a displacement

$$
\begin{aligned}
x(t) &= C\cos\omega_0 t + D\sin\,\omega_0 t \\
&= A\sin(\omega_0 t + \phi)
\end{aligned}
$$

the velocity of the mass is given by

$$
\begin{aligned}
v(t) &= \dot{x}(t) \\
&= \omega_0\left(C\cos\omega_0 t - D\sin\,\omega_0 t\right) \\
&= \omega_0 A\cos(\omega_0 t + \phi)
\end{aligned}
$$

Thus the maximum possible magnitude of the velocity is $\omega_0 A$. The general displacement and the corresponding velocity of the mass with respect to time are displayed in figure 3.



Displacement and velocity in a harmonic oscillator

Figure 3

It is clear from the figure that for a given displacement, the velocity is such that when displacement is at its maximum or minimum, the velocity is zero and when the displacement is zero, the velocity has the largest magnitude. This is physically clear. When the spring is compressed or stretched to its maximum, the particle is at rest and when the particle passes through the equilibrium point, its speed is at its maximum. Let me now solve a few examples.

Example 1: In a spring-mass system k = 16 *N/m* and m = 1 *kg* . If the mass is displaced by .05 *m* and released from rest, find its subsequent motion.

$$x(t) = C\sin\omega_0 t + D\cos\omega_0 t$$

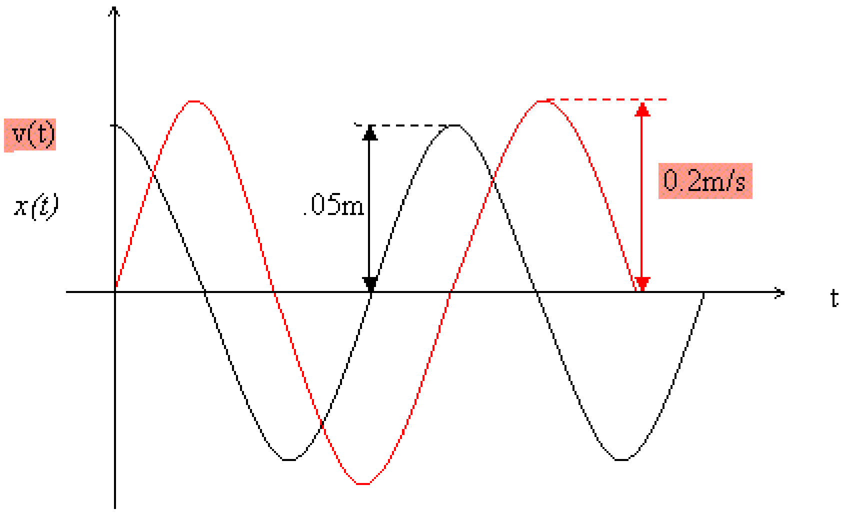
$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{16} = 4 rad/s$$

Using the initial conditions I get

$x(0) = D = 0.05m$

$$\dot{x}(0) = \omega_0 C = 0 \quad \Rightarrow \quad C = 0$$

So the solution is $x(t) = .05\cos 4t$ with the maximum speed of *0.2m/s* . The solution x(t) is plotted in figure 4. Also plotted there is the velocity *v(t)* of the mass as it performs its motion. Notice that from the x(t) curve, the velocity can be easily plotted by taking its slope.



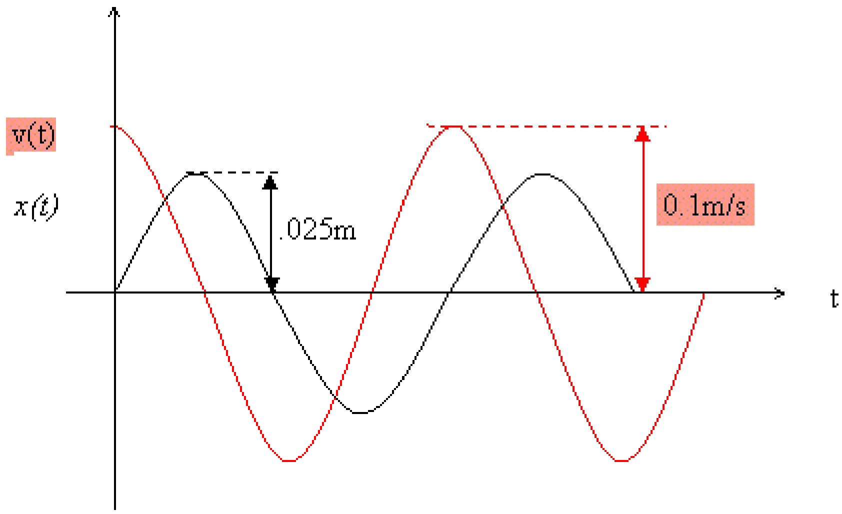
Displacement and velocity in example 1with the initial condition x(0)=0.5m and v(0)=0.

Figure 4

Let me now show you how the solution changes when the initial conditions are different. Suppose instead of pulling the mass and releasing it, I give it an initial velocity of .1m/s toward the right from the equilibrium. In that case

$$x(0) = D = 0$$
$$\dot{x}(0) = \omega_0 C = 0.1 \quad \Rightarrow \quad C = 0.025$$

So $x(t) = 0.025 \sin 4t$. Obviously the maximum speed in this case is 0.1m/s, that given in the beginning. The solution looks like shown in figure 5.



Displacement and velocity in example 1with the initial condition
x(0)=0 and v(0)=0.1m/s.

Figure 5

Third possibility of initial conditions is when I take the mass to a displacement of .05m and push it towards the equilibrium point with a speed of .1m/sec. Then

$$x(0) = D = .05$$
$$\dot{x}(0) == \omega_0 C = -.1 \quad \Rightarrow \quad C = -.1/4 = -.025$$

Thus the solution is $x(t) = -0.025\sin\omega_0 t + 0.05\cos\omega_0 t$. If we wish to express this as $x(t) = A\sin\, x(\omega_0 t + 0)$ then

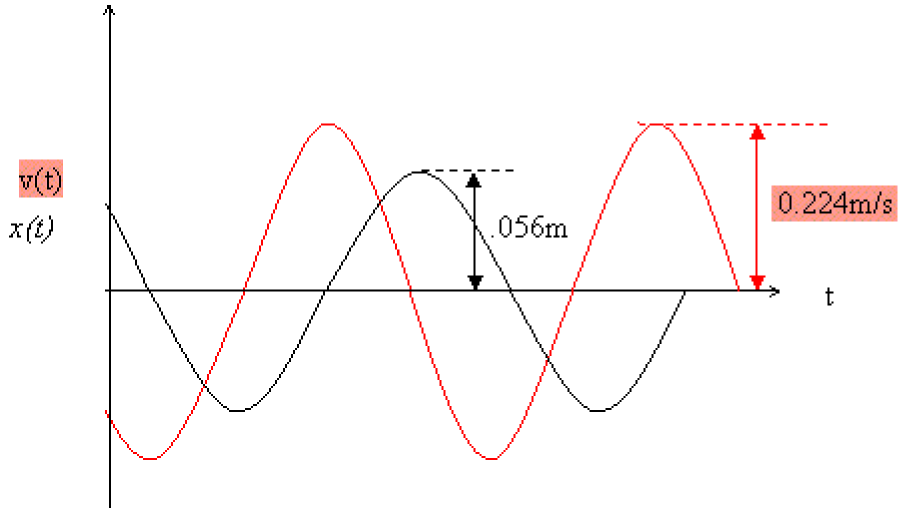
$$A = \sqrt{(.05)^2 + (.025)^2} = .056$$

and

$$\sin\phi = \frac{.05}{\sqrt{(.05)^2 + (.025)^2}} \quad ; \cos\phi = \frac{-.025}{\sqrt{(.05)^2 + (.025)^2}}$$

$$\text{and} \quad \tan\phi = -2 \quad \text{with} \quad \frac{\pi}{2} < \phi < \pi$$

This gives $\phi \approx 116^o$ and $x(t) = .056\sin(4t + 116^o)$. The maximum speed in this case is v_{max} = *4 x 0.056 = 0.224m/s* . So the graph of the motion looks like that shown in figure 6.

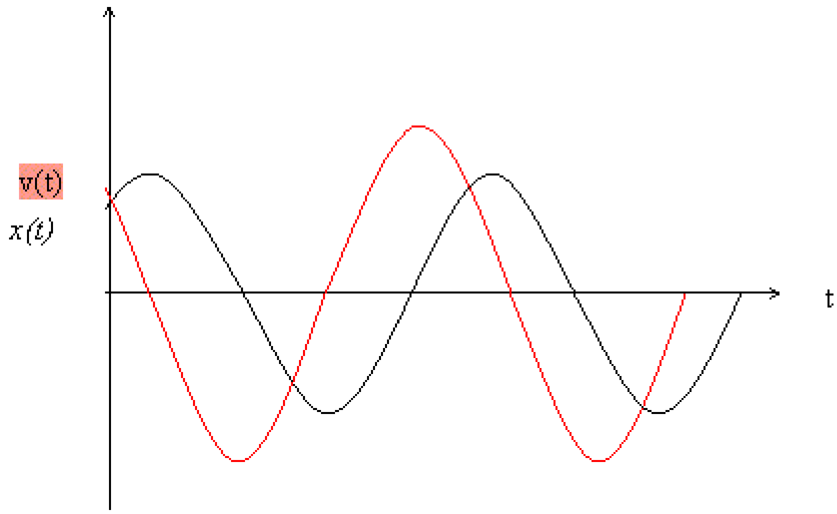


Displacement and velocity in example 1with the initial condition
$x(0)=0.05$ and $v(0)=-0.1 m/s$.

Figure 6

From the graph it is very clear that initially the speed of the particle increases in the negative direction and then the particle starts slowing down, stopping at the full compression of the spring, as is clear from the plot of its displacement.

If in the case studied just now, the mass was thrown out instead of being pushed in, it would have a positive velocity to start with but the speed would be decreasing at that moment. Then the mass will travel out to its maximum displacement and would then turn back. The general plot of displacement and velocity versus time would then look as in figure 7. I will leave it for you to work out the numbers for amplitude and initial phase.



Displacement and velocity in example 1with the initial condition
x(0)=0.05 and v(0)=+0.1m/s.

Figure 7

Example 2: In the second example I show that about any stable equilibrium point, the motion to a good degree is simple harmonic. let us take two changes of *10 μC* each at a distance of half a meter so that is a positive charge of *5 μC* is kept at the centre, its experiences no force (see figure 8). The *5 μC* charge is confined to move along the line joining the two changes. If displaced by a small distance from its equilibrium position, what kind of motion does it perform?

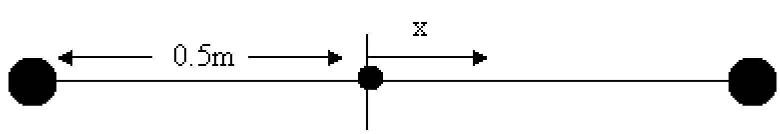


Figure 8

When the 5 μC is displaced to the right by *x*, the force on it is

$$
\begin{aligned}
F &= 9\times10^{9}\times\left(\frac{50\times10^{-12}}{(.5+x)^{2}}-\frac{50\times10^{-12}}{(.5-x)^{2}}\right)\\
&\cong -3.6x
\end{aligned}
$$

In obtaining the force above, we have used the binomial theorem to expand $(.5\pm x)^{-2}$. Since the force is proportional to the displacement and in direction opposite to it, the charge will perform simple harmonic motion.

Let me now look at some other examples, going beyond the spring-mass system.

Example 3: A disc of mass *M* and radius *R* is hanging on a will about a point on its periphery (see figure 9). If it is displaced from its initial position by small angle θ_o and released, find its subsequent motion.

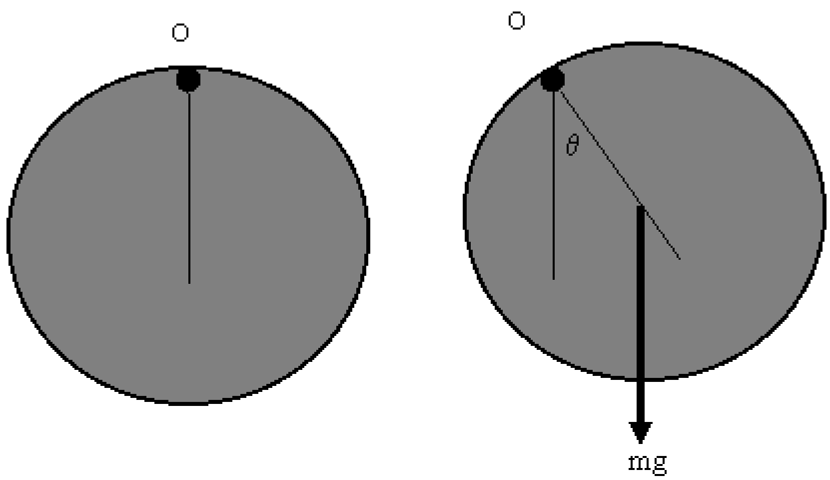


Figure 9

This is a case where a rigid body is moving under distributed forces so we use angular momentum to describe its motion. The equation of its motion therefore is

$$I_o\dot{\omega} = I_o\ddot{\theta} = -MgR\sin\theta$$
$$= -MgR\theta \ (small \ angle)$$

By transformation theorem,

$$I_o = I_{aboutCM} + MR^2 = \frac{3MR^2}{2}$$

So the equation of motion becomes

$$\ddot{\theta} + \frac{2g}{3R}\theta = 0$$

This means that in general the motion of the disc would be simple harmonic and will be given as

$$\theta(t) = C\sin\omega_0 t + D\cos\omega_0 t \quad \text{with} \quad \omega_0 = \sqrt{\frac{2g}{3R}}$$

The initial conditions in this case give $C = 0$ and $D = \theta_0$. Therefore the solution in the present case is $\theta(t) = \theta_0 \cos\sqrt{\frac{2g}{3R}}t$.

Example 4: As the final example here, let me take a particle moving in a potential $U(x) = \frac{A}{x^2} + Bx^2 \; (A > 0, B > 0)$ .The potential has a minimum at x_0 given by

$$\frac{d}{dx}\left(\frac{A}{x^2} + Bx^2\right)\bigg|_{x0} = 0$$

$$\Rightarrow -\frac{2A}{x_0^3} + 2Bx_0 = 0$$

or

$$x_o = \left(\frac{A}{B}\right)^{\frac{1}{4}}$$

You can yourself check that the second derivative at this point is positive and its value is *8B*. For very small displacements *x* about this point we have the change in the potential energy given as

$$\Delta U(x) = \frac{A}{(x_o + x)^2} + B(x_o + x)^2)$$

which by binomial theorem or the Taylor series expansion leads to

$$\Delta U(x) = U(x_o) + \frac{1}{2}(8B)x^2$$

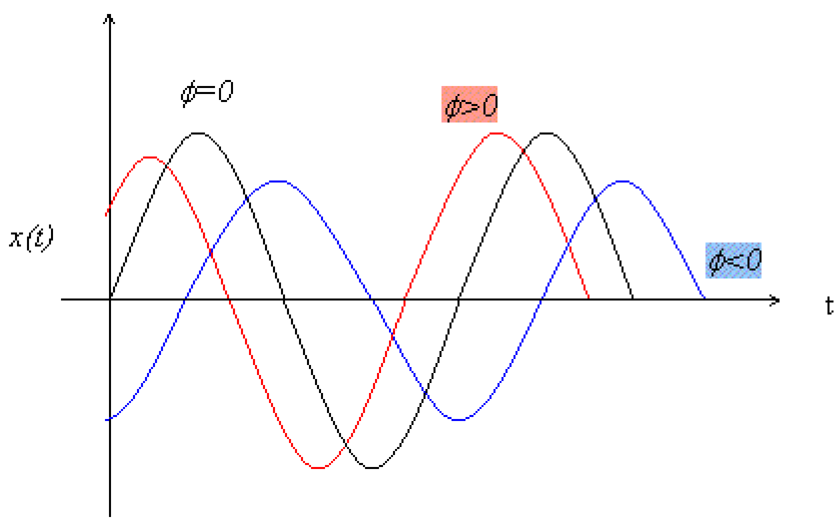
This gives an equivalent spring constant of $k=8B$ and frequency of oscillation $\omega_o = \sqrt{\frac{8B}{m}}$.

Having solved these examples I now wish to discuss a very important topic of phase and phase difference in a simple-harmonic motion. I will spend some time discussion phasor diagrams give a feel for the phase.

Phase and Phase difference in simple harmonic motion : In general the solution of a simple harmonic equation is

$$x(t) = A\sin(\omega t + \phi)$$

As mentioned earlier *A* is known as the amplitude and $(\omega t + \phi)$ as the phase. ϕ is a constant depending on the initial conditions and we call it the phase constant. Let us now see how does the motion look for different values of the phase constant ϕ. The displacement versus time plots for different signs of the phase constant are shown in figure 10.



Displacement versus time graphs for different values of phase constant. Black: ϕ=0; Red: ϕ>0 and blue is ϕ<0.

Figure 10

For Φ > 0 the motion at t = 0 begin at a value or phase angle that it would have slightly later in the $\phi = 0$ case. On the other hand, for *Φ* < 0 the motion is such that a particular displacement for the $\phi = 0$ case is reached at a later time. The motion lags behind the $\phi = 0$ motion. I leave it for you to figure out yourself how the corresponding velocities are related.

Let us now at the special case of $\phi = 180^\circ$. In this case I get

$$x(t) = A\sin(\omega t + 180^o) = -A\sin\,\omega t$$

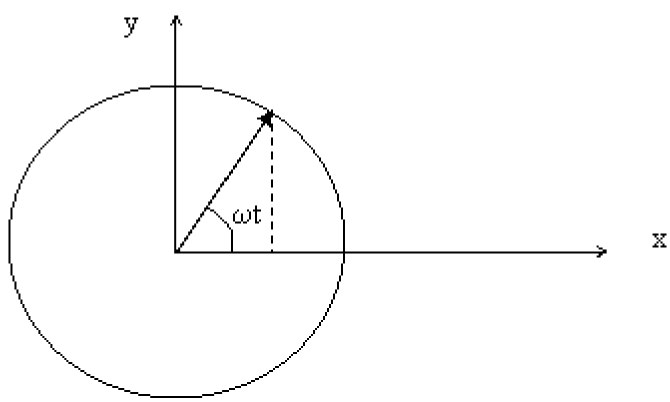
and for $\phi = -180^o$

$x(t) = A\sin(\omega t - 180^o) = -A\sin\ \omega t$

So you see that a phase difference of 180^o, whether position or negative, means the same thing. I would like you to plot the displacement versus time graph for these particular cases. For the phases in between you should see for yourself how the displacements at t = 0 are different from $\phi = 0$ case.

A good way of visualizing the simple harmonic motion is the phasor or vector diagram. I discuss that next.

Phasor or vector diagram: A nice geometric way of looking at various quantities in a simple harmonic motion is the vector or a phasor diagram. You may have seen it in your 12th grade while studying AC circuits. Let me show you how we represent $x(t) = A\cos\omega t$ in a geometric way. You see that displacement in this case is the *x* component of a vector making an angle ωt from the x-axis. Thus the displacement is represented as shown in figure 11. The motion described by $x(t) = A\cos\omega t$ is thus given by the projection of a vector of length *A*, rotating counterclockwise at a rate ω, on the x-axis.



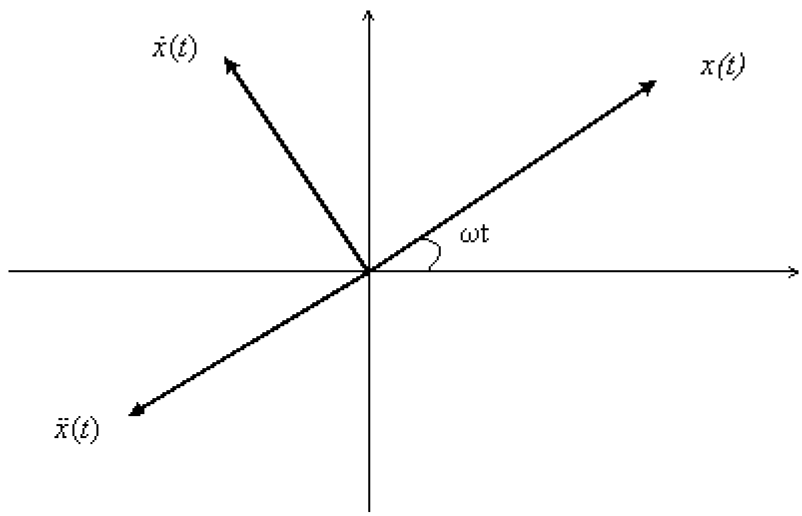
Projection of a vector $\vec{A}$ rotating counterclockwise at a rate ω on the x-axis gives $x(t) = A\cos\omega t$

Figure 11

Let us now see how the velocity $\dot{x}(t)$ and the acceleration will be represented in this scheme? The velocity and acceleration are given as

$$\dot{x}(t) = -\omega A \sin \omega t = \omega A \cos\left(\omega t + \frac{\pi}{2}\right)$$
$$\ddot{x}(t) = -\omega^2 A \cos \omega t = \omega^2 A \cos(\omega t + \pi)$$

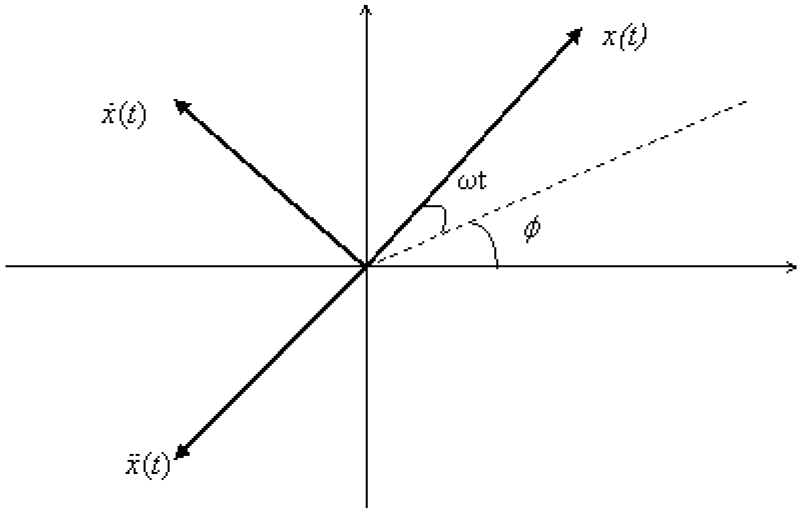
The displacement, velocity and acceleration are shown in the phasor diagram in figure 12. A general feature that we observe from this phase diagram is that the velocity vector is always $\frac{\pi}{2}$ ahead (measuring counterclockwise) of the displacement vector and the acceleration vector is at π (ahead or behind?) the displacement.



Displacement, velocity and acceleration for a simple harmonic motion shown on a phasor diagram

Figure 12

So far we have discussed the simple case of $x(t) = A\cos\omega t$. What about the general case of $x(t) = A(\cos\omega t + \phi)$. This is also equally simple. All we have to do is keep the initial position of the vector at $t = 0$ at an angle Φ from the x-axis and start rotating it from there. The velocity vector and the acceleration vector are then going to be given at $\frac{\pi}{2}$ and π from it, as discussed above. This is shown in figure 13.



Displacement $x(t) = A(\cos\omega t + \phi)$, *and the corresponding velocity and acceleration for a simple harmonic motion*

Figure 13

Recall that in the middle of this lecture I had solved a spring-mass problem with different initial conditions. I would like you to make the phasor diagram to represent the motion of the mass in many different situations like those considered above. Do not solve for x(t) to start with, just make the phasor diagram directly to see if you have got a feel for motion under different conditions.

Finally in this lecture I look at the energy of a system performing simple harmonic motion. The potential energy *U(x)* and the kinetic energies T are

$$U(x) = \frac{1}{2}kx^2 \qquad T = \frac{1}{2}m\dot{x}^2$$

The total energy *E* is of course a sum of the two. With $x(t) = A\cos(\omega t + \phi)$ this gives

$$E = \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

Since $\omega^2 = \dfrac{k}{m}$, we get

$$E = \frac{1}{2}kA^2$$

Thus the energy depends on the square of the amplitude. This makes sense because if I stretch a spring by *A*, the energy stored in it is $\frac{1}{2}kA^2$. On releasing the mass it performs oscillations of amplitude A. Thus you see that amplitude A immediately implies a total energy given above.

I have now set up all the basic concepts of simple harmonic motion. In the coming lectures I will introduce damping in the system and see how it evolves.

Harmonic oscillator II: damped oscillator

In the previous lecture, I covered some basic aspects of simple harmonic oscillations. We considered the equation

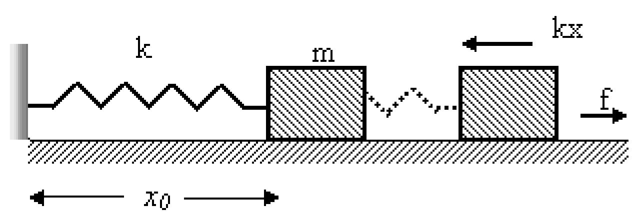
$$m\ddot{x} + kx = 0$$

and saw how its motion is described. A general solution of this equation is

$$x(t) = A\cos(\omega_o t + \phi) \text{ with } \omega_o = \sqrt{\frac{k}{m}}$$

I now make the system little more realistic and introduce damping into the system. Let us first look at what happens if we introduce friction into the system. I consider again our prototype spring-mass system and let there be a constant frictional force *f* on the mass. This force will always oppose the motion so the system will eventually come to a stop. Let us see when does it do that?

The simplest way of seeing when he system will stop is the through the consideration of energy. But I would like to solve the problem by employing the equation of motion. I will later solve it from energy considerations also. Here is one case where I will have to analyze motion step by step because as the velocity direction changes, so does the force direction. So let us pull the spring out to a distance *A* and let it move towards the equilibrium point (see figure 1).



A spring-mass system with friction. Force applied by the spring and frictional force are both shown.

Figure 1

When the block is moving towards the left, equation governing its motion will be

$$m\ddot{x} = -kx + f$$

In the above, the frictional force *f* sign is positive because the mass is moving in the negative *x* direction and therefore the frictional force is in positive *x* direction. This equation can be recast into the form

$$m\ddot{x} + \omega_0^2 x = \frac{f}{m} \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

We have encountered such kind of equation earlier. It has a homogeneous part $\ddot{x} + \omega_0^2 x = 0$ and an inhomogeneous term on the right-hand side. So the general solution is

$$x = x_{\text{homogeneous}} + x_{\text{inhomogeneous}}$$

where

$$x_{\text{homogeneous}} = C\cos\omega_o t + D\sin\,\omega_o t$$
$$x_{\text{inhomogeneous}} = \frac{f}{\left(m\omega_o{}^2\right)} = \frac{f}{k}$$

Thus

$$x(t) = C\cos\omega_o t + D\sin\,\omega_o t + \frac{f}{k}$$

With the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, the solution is

$$x(t) = \left(A - \frac{f}{k}\right)\cos\omega_o t + \frac{f}{k}$$

This is the solution when the block is moving to the left. Since

$$\dot{x} = -\left(A - \frac{f}{k}\right)\omega_o\sin\,\omega_o t$$

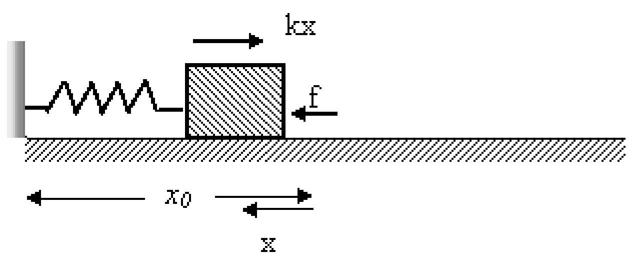
so the block will come to a stop when $(\omega_o t) = \pi$. At that time

$$x(t) = -A + \frac{f}{k}\omega_o \cos\omega_o t + \frac{f}{k}$$
$$= -A + \left(\frac{2f}{k}\right)$$

So by the time the block comes to a stop it has lost $\frac{2f}{k}$ distance from its amplitude. And this loss is irrespective of the distance from where the block starts its motion from. This should then also happen when the block starts coming back. Let us find that out. On its way back (see figure 2), the block follows the equation

$$m\ddot{x} = -kx - f$$

Notice that the sign of the friction force is now negative. This is because now the block is traveling to the right and therefore the friction force acts towards the left (see figure 2).



A spring-mass system with friction when the spring is compressed. Force applied by the spring and frictional force are both shown.

Figure 2

Now we have to solve this equation with the initial condition that

$$x(0) = -A + \left(\frac{2f}{k}\right) \quad \text{and} \quad \dot{x}(0) = 0$$

I leave it as an exercise for you to get the solution. It is

$$x(t) = -\left(A - \frac{3f}{k}\right)\cos\omega_o t - \frac{f}{k}$$

The corresponding velocity is proportional to $\omega_0 t$, and therefore goes to zero again after a time interval of $\frac{\pi}{\omega_0}$. At that time $x = \left(A - \frac{4f}{k}\right)$. Thus every half time the block goes from one extreme to the other, it loses a distance of $\left(\frac{2f}{k}\right)$, and in each cycle it loses a distance of $\left(\frac{4f}{k}\right)$. Question is how many cycles does the block complete before it comes to a stop. The block stops when its final displacement is $\frac{f}{k}$. If it completes *n* cycles before that, we have

$$A - n\frac{4f}{k} = \frac{f}{k} \quad \Rightarrow \quad n = \frac{k}{4f}\left(A - \frac{f}{k}\right)$$

The same result can also be obtained, as I said earlier, by energy methods. If stretched by *A* the total energy of the system is $\frac{1}{2}kA^2$. Let us say that before stopping, the block it compresses the spring by A_1. Then its energy will be $\frac{1}{2}kA_1^2$. The loss in the energy is caused by friction. Thus

$$\frac{1}{2}kA^2 - \frac{1}{2}kA_1^2 = \text{energy loss due to friction}$$

The total distance moved by the block is $(A+A_1)$ and so the energy lost against friction is $f(A+A_1)$. Thus the equation transforms to

$$\frac{1}{2}kA^2 - \frac{1}{2}kA_1^2 = f(A + A_1)$$

and gives

$$(A - A_1) = \frac{2f}{k}$$

which is the same loss in amplitude over half a cycle as obtained earlier. The rest of the analysis is the same as done earlier.

Having dealt with the constant friction case, we now consider the most common example of damped oscillations. This is the oscillator where damping force is proportional to the velocity i.e.,

$$\begin{aligned} F_{retardation} &= -bv \\ &= -b\dot{x} \end{aligned}$$

In this case, the equation of motion is

$$m\ddot{x} = -kx - b\dot{x} \quad \text{or} \quad m\ddot{x} + b\dot{x} + kx = 0$$

Writing $\frac{b}{m} = \gamma$ we get

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

This is the equation for a damped oscillator. The equation is homogeneous in *x* so we assume a solution $x(t) = e^{\lambda t}$ and substitute it in the equation to get.

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0$$

which gives

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

So the general solutions are

$$\exp\left(-\frac{\gamma}{2}t + \sqrt{\frac{\gamma^2}{4} - \omega_0^2 t}\right) \quad \text{and} \quad \exp\left(-\frac{\gamma}{2}t - \sqrt{\frac{\gamma^2}{4} - \omega_0^2 t}\right)$$

Except in the case when $\frac{\gamma^2}{4} = \omega_0^2$ (we will deal with it later) the behavior of the solution depends on the relation magnitude of γ and ω_0. Let us first consider the case when $\frac{\gamma^2}{4} > \omega_0^2$. In that case

$$\lambda = -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = -\lambda_1 \quad \text{and} \quad \lambda = -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = -\lambda_2$$

The general solution then is

$$x(t) = -Ce^{-\lambda_1 t} + De^{-\lambda_2 t} \quad \text{where} \quad \lambda_2 > \lambda_1$$

This is known as a heavily damped oscillator. The coefficients *C* and *D* depend on the initial conditions. For example if I stretch the spring to a distance *A* and release the block, let us see what happen in this case. By initial conditions

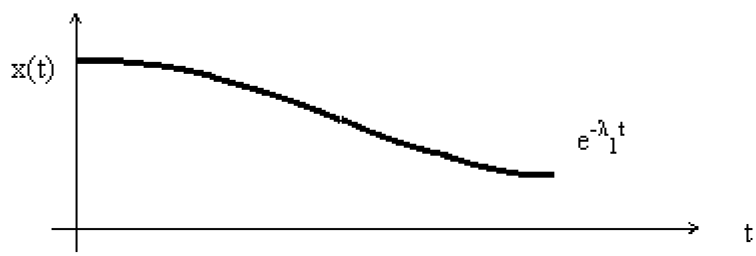
$$x(0) = A = C + D$$

$$\dot{x}(0) = 0 \Rightarrow -\lambda_1 C - \lambda_2 D = 0 \text{ or } D = -\frac{\lambda_1}{\lambda_2} C$$

This leads to

$$x(t) = A\left[\frac{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1}\right]$$

A t → ∞ this solution behaves like $x(t) = \frac{A\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}$. The general solution is displayed in figure 3.



Displacement vs. time for a heavily damped oscillator

Figure 3

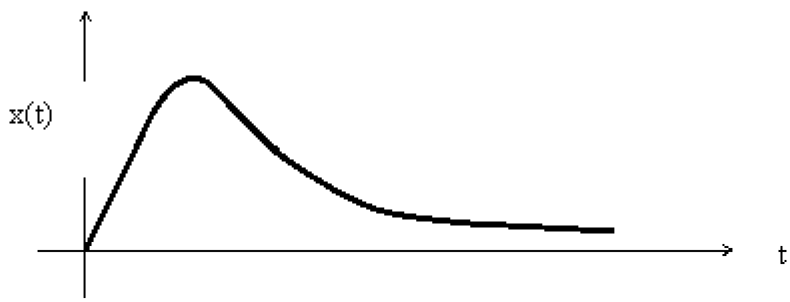
It is clear from the figure that there are no oscillations in this case the block slowly comes to rest at *x = 0* , i.e. the equilibrium point. I now explore another situation. Suppose we give an implies (speed *v*) at *t = 0* then the boundary conditions are

$$C + D = 0 \qquad \text{and} \qquad -\lambda_1 C - \lambda_2 D = v$$

Thus the solution would be

$$x(t) = \frac{v}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

In this case the distance versus time graph looks as shown in figure 4.



Displacement vs. time for a heavily damped oscillator when the block is given an impulse at the equilibrium point.

Figure 4

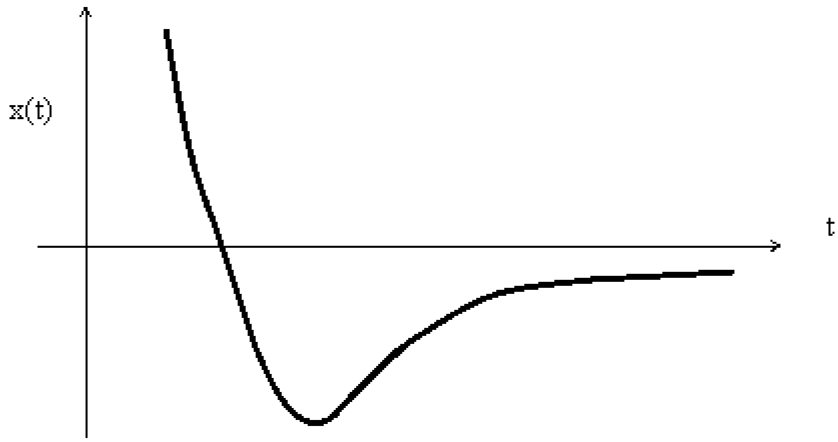
The figure clearly shows that the block goes out to a maximum distance and then comes back and stops at the equilibrium point. So in both the cases studied above the mass does not cross the equilibrium point. Next I ask: what if we stretch the mass out to a distance *A* and give it an initial impulse from that point (in negative direction). Then the initial conditions will b

$C + D = A \qquad \text{and} \qquad -\lambda_1 C - \lambda_2 D = -v$

Solution in this case comes out to be

$$x(t) = -\frac{v}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + \frac{A}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}\right)$$

The solution is plotted schematically in figure 5.



Displacement vs. time for a heavily damped oscillator when the block is given an impulse after stretching the spring .

Figure 5

It is clear that in this case the particle moves towards the equilibrium point, crosses it, goes a distance and comes back. However on its way back it slowly comes to rest at the equilibrium point and does not cross it. So in heavy damping cases, the block passes the equilibrium point at most once and its distance decays exponentially as $e^{-\lambda_1 t}$.

To summarize, I have covered three cases for the heavy damping situation and got

(i) Spring stretched and block released $x(t) = \dfrac{A}{\lambda_2 - \lambda_1}[\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}]$

(ii) The block given an initial positive velocity at equilibrium $x(t) = \dfrac{v}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$

(iii) Spring stretched out and the block given a velocity in the negative direction

$$x(t) = -\frac{v}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + \frac{A}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}\right)$$

I would now like to tell you about the case when $\lambda_1 = \lambda_2$. This is known as the critically damped case. Obviously this situation arises when $\dfrac{\gamma^2}{4} = \omega_0^2$. I can easily find solutions for such case if I take the limit $\lambda_1 \to \lambda_2$ in the cases of heavy damping just studied. Please note that I cannot straightaway take $\lambda_1 = \lambda_2$ in the expressions above because I am dividing by

$(\lambda_2 - \lambda_1)$. Taking the limit gives for the three cases studied above

(i) $x(t) = Ae^{-\lambda_1 t}(1+\lambda_1 t)$

(ii) $x(t) = vte^{-\lambda_1 t}$

(iii) $x(t) = (A - vt)e^{-\lambda_1 t}$

As remarked above, the cases we have just discussed correspond to critical damping. In this situation $\frac{\gamma^2}{4} = \omega_0^2$ and $\lambda_1 = \lambda_2 = -\frac{\gamma}{2}$. Mathematically, in this case there is only one solution ($e^{-\frac{\gamma}{2}t}$) that we get from the equation for λ because of its double root. The other solution is found to be $te^{-\frac{\gamma}{2}t}$. That is precisely what we have found by taking appropriate limit.

Critically damped system and used when we want a system to return to its equilibrium position after receiving an impulse, although one is tempted to say that use a heavily damped system for this purpose. I would like you to understand this by carrying out the following exercise.

Exercise : The block on a damped spring-mass system is given an initial velocity v from equilibrium. Given a damping coefficient γ, plot the distance versus time graph for the critically and heavily damped cases. For ease of calculation take the heavy damping to be very large so that $\frac{\gamma^2}{4} >> \omega_0^2$ and make appropriate approximations.

Having discussed the heavily and critically damped systems, we move on to lightly damped system. In such systems $\frac{\gamma^2}{4} < \omega_0^2$ so that

$$\lambda_1 = \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \frac{\gamma}{2} - i\,\omega_1$$
$$\lambda_2 = \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \frac{\gamma}{2} + i\,\omega_1$$

So the general solution is

$$x(t) = e^{-\frac{\gamma}{2}t}\left(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right)$$

Or equivalently

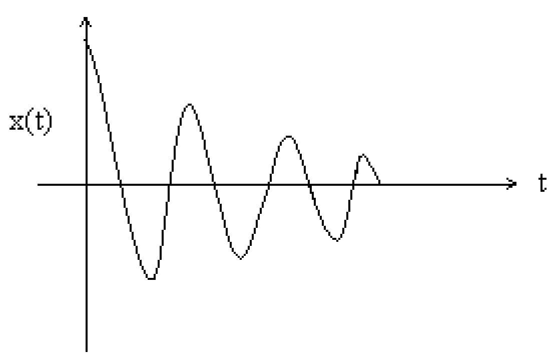
$$x(t) = e^{-\frac{\gamma}{2}t}\left(C\cos\omega_1 t + D\sin\omega_1 t\right)$$

In case when $\frac{\gamma^2}{4} << \omega_0^2$, it is called very light damping and in such case $\omega_1 \approx \omega_0$.

Let us now take a particular can when the block is stretched to distance *A* and is released from rest. I leave the details of the solution to be worked out by you. Here I give the final answer which is

$$x(t) = \frac{A\omega_0}{\omega_1}e^{-\frac{\gamma}{2}t}\cos(\omega_1 t - \phi) \quad \text{where} \quad \tan\phi = \frac{\gamma}{2\omega_1}$$

This solution is plotted schematically in figure 6. Notice how the maximum distance reached by the block decreases with time.



Displacement versus time in a damped oscillator

Figure 6

When we consider light damping, generally we are dealing with cases where we want the decay to be small. Thus within the time that the motion decays, there are many-many oscillations. Thus we can then write the displacement as

$$x(t) = \frac{A\omega_0}{\omega_1} e^{-\frac{\gamma}{2}t} \cos(\omega_1 t - \phi)$$
$$= Ae^{-\frac{\gamma}{2}t} \cos \omega_0 t$$

because $\frac{\gamma^2}{4} << \omega_0^2$ implies that $\omega_1 \approx \omega_0$ and $\phi \approx 0$. The equation above is interpreted as the oscillation taking place with frequency w 0 with time-dependent amplitude $Ae^{-\frac{\gamma}{2}t}$. Mathematically what this means is that $\frac{1}{\gamma} >> \frac{1}{\omega_0}$ so there are two time scales in the problem. Let me now talk about the energy of the system. Since the amplitude is decreasing with time, the system is obviously losing energy. I want to calculate the rate of energy loss in the system.

First, there and many oscillations over the time interval of $\frac{1}{\gamma}$, which is also a very large time span. Further, the decay of the amplitude is very small over a few periods. This allows us to talk in terms of the average energy of the system. What it means is the energy averaged over a few cycles around a given instant. I now calculate it.

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Now use $x(t) = Ae^{-\frac{\gamma}{2}t} \cos \omega_0 t$ to calculate this energy. It gives

$$E(t) = \frac{1}{2}m\left[\frac{\gamma^2}{4}A^2\cos^2(\omega_0 t) + \omega_0^2 A^2 \sin^2(\omega_0 t) + \gamma\omega_0 A^2 \sin(\omega_0 t)\cos(\omega_0 t)\right]e^{-\gamma t}$$
$$+\frac{1}{2}kA^2\cos^2(\omega_0 t)e^{-\gamma t}$$

Now taking an average over a few cycles under the approximation that the exponentially decaying term be treated as roughly a constant over these cycles and neglecting the term proportional to γ^2, I get

$$\langle E(t) \rangle = \frac{1}{2}kA^2 e^{-\gamma t}$$

where angular brackets denote the average energy. So the average energy decays exponentially for a lightly damped oscillator.

I now define the quality factor or Q for an oscillator. As mentioned earlier, we are interested in systems where $\gamma << \omega_0$; it is in such cases only that talking about Q makes sense. Q is defined as

$$Q = \frac{E_{stored}}{E_{dissipated} / radian}$$
$$= \frac{\omega_0}{\gamma}$$

High Q value for an oscillator means that there is very low leakage compared to the store energy.

Finally I summarize the lecture by telling you that we have covered the cases of heavy, critical and light damping in this lecture. You must have noticed that I have made a lot of graphs in this and the previous lecture. Please do that when you solve a problem. It will give you a feel for the system

Harmonic oscillator III: Forced oscillations

In the previous two lectures, you have learnt about free harmonic oscillator and damped harmonic oscillator. In this lecture we study what happens when a harmonic oscillator is subjected to a force. The simplest case is when an oscillator is subjected to a constant force F. In that case nothing much takes place except that the equilibrium point gets shifted by *(F/k)*. You see an example of it when a mass is attached to a vertical spring. Mathematically we write

$$m\ddot{x} + kx = F$$
$$\text{or} \quad m\ddot{x} + kx - F = 0$$

This can be written as

$$m\ddot{x} + k\left(x - \frac{F}{k}\right) = 0$$

for an undamped oscillator and

$$m\ddot{x} + \gamma\dot{x} + k\left(x - \frac{F}{k}\right) = 0$$

for a damped oscillator. Define a new variable $y = x - \frac{F}{k}$ so that the equation reads (I write only the undamped oscillator equation)

$$m\ddot{y} + ky = 0$$

This is the equation you are well familiar with. From its solution, that for *x* is written as

$$x = C\cos\omega_0 t + D\sin\ \omega_0 t + F/k$$

So the mass oscillates about $X = F/k$. I now take up an oscillator subjected to a time-dependent force.

A general time-dependent force *F(t)* can always be decomposed into its Fourier components like $F(t) = \sum_n F_n \cos(n\omega t)$ so generally we study an oscillator subjected to a force of the form. $F(t) = F\cos\omega t$, where $\omega \neq \omega_o$ and *F* is the amplitude of the force. Let me start by first studying the motion of an undamped oscillator under such a force.

The equation of motion for an undamped oscillator under a time-periodic force is

$$m\ddot{x} + kx = F\cos(\omega t)$$

or equivalently

$$\ddot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega t)$$

The general solution is a combination of homogeneous part of the equation and a particular solution x_p. Thus

$$x(t) = \cos\omega_0 t + D\sin\ \omega_0 t + x_p$$

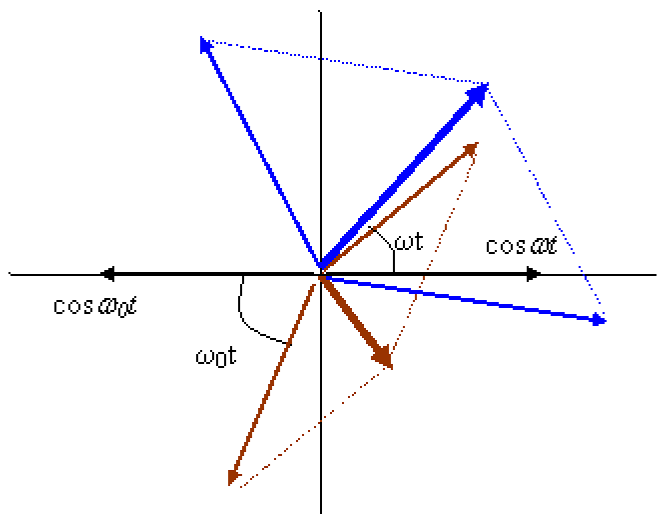
Here you can check that

$$x_p = \frac{F/m}{(\omega_o{}^2 - \omega^2)}\cos\omega t$$

Let me start the oscillator from rest at equilibrium. It starts moving because of the applied force. The initial conditions then are $x(0) = 0$ and $\dot{x}(0) = 0$. Under these conditions the solution comes out to be

$$x(t) = \frac{F}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t)$$

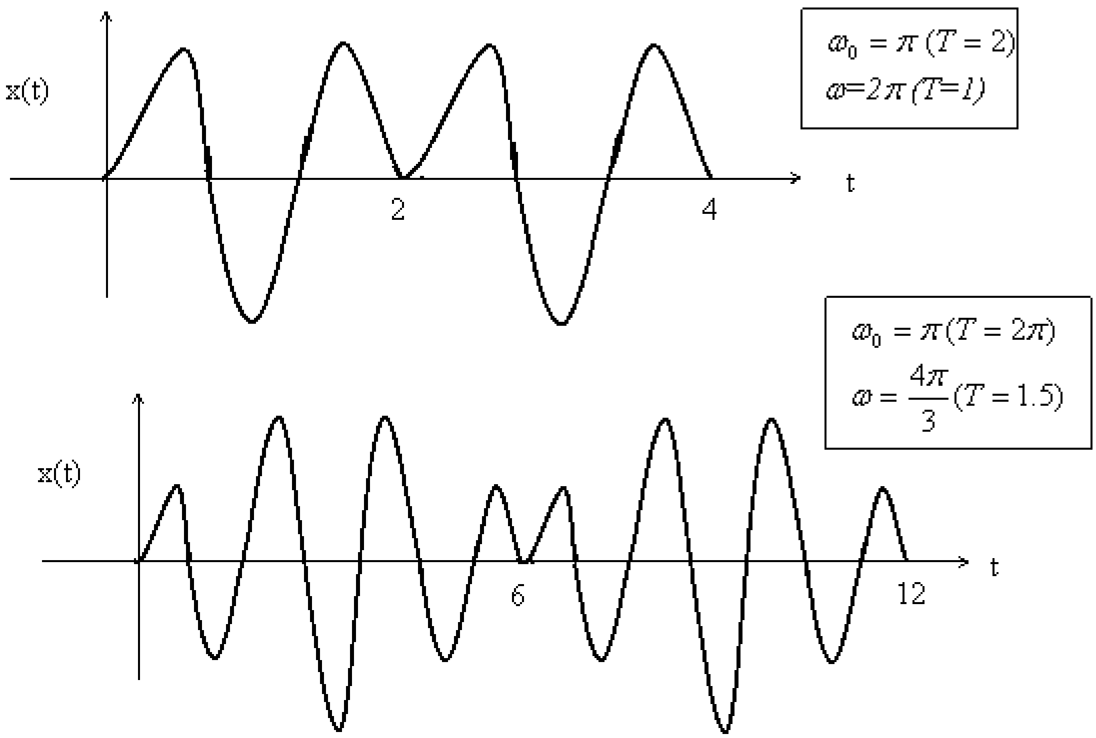
So the general solution is a combination of motion of two frequencies. The resulting motion can be represented on a phasor diagram by adding the two motions vectorially. This shown at *t = 0* and two other different times in figure 1.



Motion of an undamped oscillator shown at t=0 (black) and two other different times, shown by brown and blue. The net displacement is shown by thick arrows.

Figure 1

As is clear from the figure, at t = 0, the net displacement is zero. As the time progresses, the displacement changes with the length of the rotating vector also changing with time. As an illustrative example, I take the frequency $\omega_0 = \pi (T_0 = 2)$, and two different frequencies, $\omega = 2\pi$ and $\omega = \dfrac{4\pi}{3}$ for the force. The resulting solutions are shown in figure 2.



Forced oscillations of an undamped oscillator of frequency $\omega_0 = 2\pi$.
Solutions are shown for two different frequencies of the applied force.

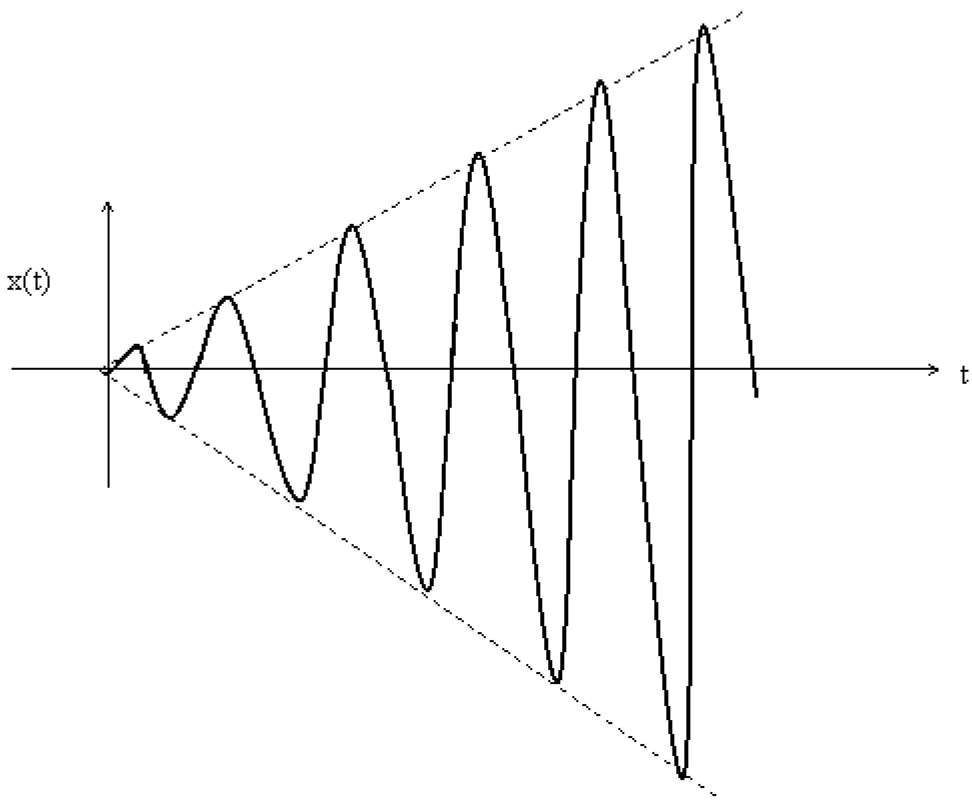
Figure 2

So you see from the figure above that the maximum displacement of oscillations keeps changing. This is what I had inferred from the phasor diagram also. The motion is still periodic and reminds us of the phenomena of beats.

Interesting is the case when $\omega = \omega_o$. However, I cannot put it directly in the formula become we are dividing by $(\omega_o{}^2 - \omega^2)$. So we have to take the limit $\omega \to \omega_0$. Let me substitute in the formula $\omega = (\omega_o - \Delta)$ or $\omega = (\omega_o + \Delta)$ and take $\Delta \to 0$. This leads to

$$x(t) = \frac{Ft}{2m\omega_0} \cdot \sin\, \omega_o t$$

Thus the displacement keeps on increasing with time oscillating with the frequency of the oscillator. This is the phenomena of resonance. The corresponding plot of displacement is shown in figure 3.



Forced oscillations of an undamped at resonance

Figure 3

Having discussed forced oscillations for undamped oscillator, we now move on to study a damped oscillator moving under the influence of a periodic force. The equation of motion then is

$$m\ddot{x} + bx + kx = F\cos(\omega t)$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega t)$$

As earlier, the general solution of this equation is going to the sum of the homogenous and inhomogeneous part. So

$$x(t) = e^{-\frac{\gamma}{2}t}[C\cos\omega_1 t + D\sin\ \omega_1 t] + x_{particular}$$

As the time progresses $e^{-\frac{\gamma}{2}t}$ will make the homogeneous solution die down so finally the only solution remaining will be

$$x(t) = x_{particular}(t)$$

This is known as the steady state solution. Obviously it does not depend on the initial conditions. Let us now find this solution.

For the equation of motion

$$\ddot{x} + \gamma\dot{x} + \omega_o{}^2 x = \frac{F}{m}\cos(\omega t)$$

I assume a steady state solution of the form $A\cos\omega t.$ But when substituted in the equation, this will give rise to a term containing $\sin\ \omega t$ because of $\gamma\dot{x}$ in the equation. So a general solution should be of the form.

$$x(t) = A\cos\omega t + B\sin\ \omega t]$$

When substituted in the equation, this leads to

$$\cos\omega t\left[(\omega_o{}^2 - \omega^2)A + \gamma\omega B\right] + \sin\ \omega t\left[(\omega_o{}^2 - \omega^2)B - \gamma\omega A\right] = \frac{F}{m}\cos\omega t$$

$$\Downarrow$$

$$(\omega_o{}^2 - \omega^2)B - \gamma\omega A = 0 \quad \text{and} \quad (\omega_o{}^2 - \omega^2)A + \gamma\omega B = \frac{F}{m}$$

These equations give

$$A = \frac{(\omega_o{}^2 - \omega^2)\,{}^F\!/_m}{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2} \quad \text{and} \quad B = \frac{\gamma\omega\,{}^F\!/_m}{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}$$

So the general solution is

$$x(t) = \frac{(\omega_o{}^2 - \omega^2)\, F/m}{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2} \cos \omega t + \frac{\gamma\omega\, F/m}{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2} \sin\ \omega t$$

$$= \frac{F/m}{\sqrt{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}} \left[\frac{(\omega_o{}^2 - \omega^2)}{\sqrt{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}} \cos \omega t + \frac{\gamma\omega}{\sqrt{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}} \sin\ \omega t \right]$$

$$= \frac{F/m}{\sqrt{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \phi)$$

where

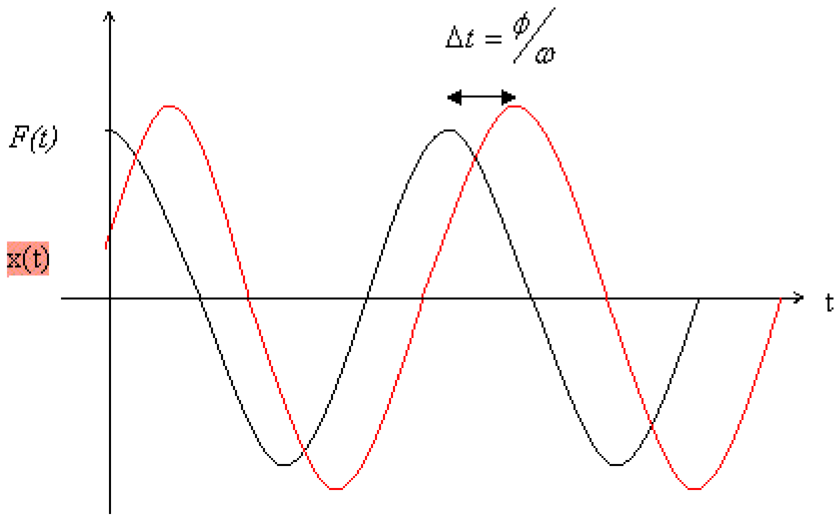
$$\sin\ \phi = \frac{\gamma\omega}{\sqrt{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}} \qquad \cos \phi = \frac{(\omega_o{}^2 - \omega^2)}{\sqrt{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

and $\qquad \tan \phi = \dfrac{\gamma\omega}{(\omega_o{}^2 - \omega^2)}$

Thus after reaching steady state, the displacement lags behind the applied force by an angle ϕ with $\tan \phi = \dfrac{\gamma\omega}{(\omega_o{}^2 - \omega^2)}$ and oscillates with an amplitude

$$A = \frac{F/m}{\sqrt{(\omega_o{}^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

The oscillation frequency of steady-state solutions is obviously equal to the frequency of the applied force. A typical displacement and its shift with respect to the applied force are shown in figure 4.



Applied force (black) and steady-state displacement (red) of an oscillator. Displacement lags behind the applied force by an angle ϕ

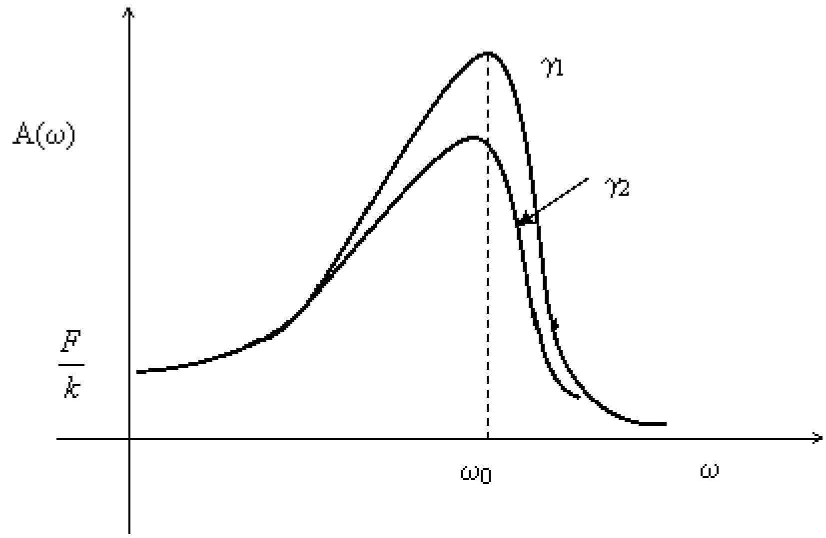
Figure 4

As far as getting the steady state solution for a forced damped oscillator is concerned, we are done. What we need to do now is to analyze the solution in different situations.

First of all we notice that irrespective of whether the system is lightly damped or heavily damped, it will always oscillate under an applied time-periodic force. Let us first consider the case of light damping and see how the amplitude varies with the applied frequency. The amplitude as a function of ω is given as

$$A(\omega) = \frac{F/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

This amplitude goes to $\frac{F}{m\omega_o^2} = \frac{F}{k}$ as $\omega \to 0$. This is nothing but the stretch of the spring under a constant force. For very large frequencies $A \approx \frac{F}{m\omega^2}$. In between the amplitude has a maximum at $\omega \approx \omega_o$ as is easily seen. So in this case, the amplitude as a function of frequency looks as shown in figure 5 for two different values of γ.

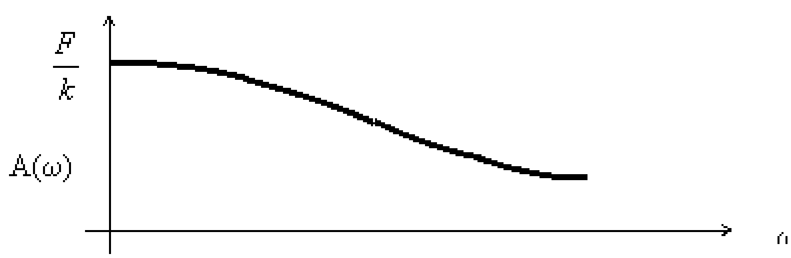


Amplitude as a function of frequency of applied force for two different values of γ with $\gamma_1 < \gamma_2$.

Figure 5

It is clear from the figure that the amplitude is maximum around $\omega = \omega_o$ which reminds us of the phenomenon of resonance for undamped oscillator. For large γ values the peak shifts to the left (lower frequency).

For heavy damping ($\gamma > 2\omega_o$) we do not see any amplitude maximum near $\omega = \omega_o$ but the system has large amplitude for low frequencies. A schematic plot of amplitude as a function of frequency looks like figure 6. It is evident that only for low frequencies the system oscillates with reasonable amplitude.



Amplitude as a function of frequency of applied force for a heavily damped oscillator.

Figure 6

What about the phase of the system with respect to the applied force? I leave this as an exercise for you to plot the phase of displacement as a function of frequency.

Next I discuss how much power is absorbed by the system to maintain its oscillations.

Power absorption in a forced damped oscillator : Since a damped system has a retardation force opposing its motion, it dissipates energy. For it to maintain a steady-state the applied force constantly supplies energy to it. It is this power that I now calculate. Power given to the system is $Fv = F\dot{x}$ since I am considering a one dimensional system. Otherwise I would have taken the dot product between the force and the velocity. The calculation proceeds as follows

$$
\begin{aligned}
P &= F\dot{x} \\
&= -F.\cos\omega t \times \frac{F/m}{\sqrt{(\omega_o^2-\omega^2)^2+\gamma^2\omega^2}}\,\omega\sin(\omega t-\phi) \\
&= -\frac{F^2\omega}{m\sqrt{(\omega_o^2-\omega^2)^2+\gamma^2\omega^2}}\left[\cos\omega t\sin\,\omega t\cos\phi-\cos^2\,\omega t\sin\,\phi\right]
\end{aligned}
$$

Since the average of $\cos^2\theta$ or $\sin^2\theta$ over a cycle is ½ and that of $\cos\theta\sin\theta$ zero, the average the last expression with respect to time over one cycle gives

$$P = \frac{F^2\omega}{2m\sqrt{(\omega_o^2-\omega^2)^2+\gamma^2\omega^2}}\sin\,\phi$$

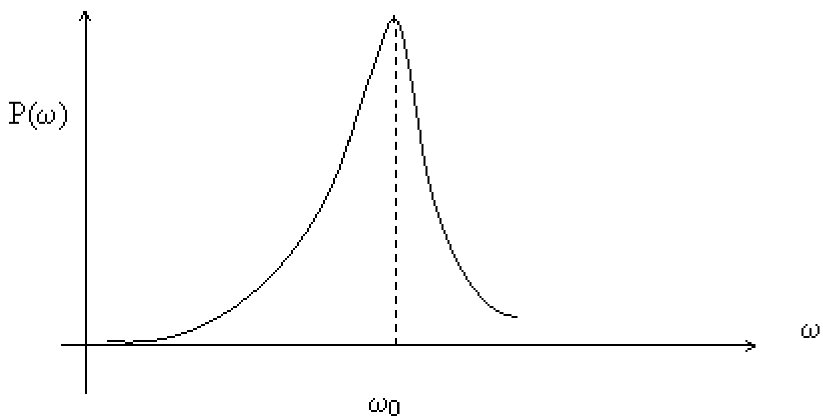
This is the average power being supplied to the system to maintain its steady-state. The same can also be obtained by realizing that in steady-state the power given to the system is the same as power dissipated by it. Power dissipated is the drag force $\left(F_{drag} = -bv = -\gamma m v\right)$ times the velocity. This is therefore calculated as follows:

$$
\begin{aligned}
P &= -bv^2 \\
&= -\gamma m\cdot\frac{F^2/m^2}{[(\omega_o^2-\omega^2)^2+\gamma^2\omega^2]}\,\omega^2\sin^2(\omega t-\phi)
\end{aligned}
$$

Taking its time average over a cycle then gives the average dissipated power

$$P = \frac{-F^2\gamma\omega^2}{2m[(\omega_o^2-\omega^2)^2+\gamma^2\omega^2]}$$

which is the same result as obtained above. The negative sign shows that this is the energy lost, and produced heat due to the friction in the system. Since the amplitude of the motion is largest when the force has a frequency close to the natural frequency of a system, it is expected that the power loss will also be maximum near that frequency. I have plotted the power dissipated in a forced damped harmonic oscillator in figure 7.



Power dissipation in a forced damped oscillator as a function of frequency of the applied force.

Figure 7

The curve peaks at ω_0 so the power absorption is indeed maximum at the resonance frequency.

Finally I relate the Q factor of a damped oscillator with the power versus frequency curve given above. To do this let us see at what frequency does the power absorption is ½ of its peak value. The calculation, in which we make the frequency-dependent factor in the expression for power dimensionless and equate it to ½, is given below

$$\frac{\gamma^2\omega^2}{[(\omega_o{}^2-\omega^2)^2+\gamma^2\omega^2]}=\frac{1}{2}$$
$$2\gamma^2\omega^2=(\omega_o{}^2-\omega^2)^2+\gamma^2\omega^2$$
$$\text{or }\ \omega_o{}^2-\omega^2=\pm\gamma\omega$$

Solving this equation for the frequency ω under the approximation of light-damping gives

$$\omega=\omega+\frac{\gamma}{2}\quad\text{and}\quad\omega=\omega_0-\frac{\gamma}{2}$$

The frequency width from $\omega_o - \frac{\gamma}{2}$ to $\omega_o + \frac{\gamma}{2}$ is known as full width at half maximum (FWHM) and its value is γ. Thus the quality factor can also be interpreted as

$$Q = \frac{\omega_o}{\gamma} = \frac{resonance\ frequency}{full\ width\ half\ maximum}$$

This pretty much sums up what I want to tell you about forced oscillations. I want to point out that we have focused here strictly on the steady-state solutions for the damped oscillator. However, before steady-state is reached, the system goes through transient motion, which is also important to understand in designing of systems.

This lecture brings us to the close of our discussion on harmonic oscillators.